Finance, Economics and Nature: Stochastic Optimal Control under Shifts, Switches and Impulses

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Stock Price Dynamics

\[ dX_t = a(X_t, t)dt + b(X_t, t)dW_t \]

This equation represents the dynamics of stock price changes, where \( X_t \) is the stock price at time \( t \), \( a(X_t, t) \) is the drift term, \( b(X_t, t) \) is the diffusion term, and \( W_t \) is a Wiener process representing random shocks.

The graphs show the stock price dynamics of DAX and logarithmic DAX over time.
NIG Lévy Asset Price Model

Normal Inverse Gaussian Process (NIG) is the subclass of generalized hyperbolic laws and has the following representation:

\[ f_{NIG}(x; \alpha, \beta, \delta, \mu) = \alpha \exp \left\{ \delta \sqrt{\alpha^2 - \beta^2} + \beta(x - \mu) \right\} \frac{K_1(\alpha \sqrt{\delta^2 + (x - \mu)^2})}{\sqrt{\delta^2 + (x - \mu)^2}}, \]

where

\[ K_1(y) = \frac{y}{4} \int_0^\infty \exp(t + \frac{y^2}{4t}) t^{-2} dt. \]
Stock Price Dynamics

NIG Lévy Asset Price Model

DAX Empirical and Simulation

Emp. Sim.
Definition: A cadlag adapted processes

\[ L = (L_t \mid t \geq 0) \]

defined on a probability space \((\Omega, F, P)\) is said to be a **Lévy processes**, if it possesses the following properties:
Lévy Processes

(i) \( P( L_0 = 0 ) = 1. \)

(ii) For \( 0 \leq s \leq t, \) \( L_t - L_s \) is independent of \( F_s, \)
i.e., \( L \) has independent increments.

(iii) For any \( 0 \leq s \leq t, \) \( L_t - L_s \) is equal in distribution to \( L_{t-s} \)
(the distribution of \( L_{t+s} - L_s \) does not depend on \( t \));
\( L \) has stationary increments.

(iv) For every \( s,t \geq 0 \) and \( \varepsilon > 0, \)
\[ \lim_{s \to t} P( | L_t - L_s | > \varepsilon ) = 0, \]
i.e., \( L \) is stochastically continuous.
There is strong interplay between

Lévy processes and infinitely divisible distributions.

**Proposition:** If \( L \) is a Lévy processes, then \( L_t \) is infinitely divisible for each \( t > 0 \).

**Proof:** For any \( n \in \mathbb{N} \) and any \( t \geq 0 \):

\[
L_t = L_{t/n} + \left( L_{2t/n} - L_{t/n} \right) + \ldots + \left( L_t - L_{(n-1)t/n} \right).
\]

Together with the stationarity and independence of increments we conclude that the random variable \( L_t \) is infinitely divisible.
Moreover, for all $u \in \mathbb{R}$ and all $t \geq 0$ we define

$$\Psi_t(u) := \ln E \left[ e^{iuL_t} \right];$$

hence, for rational $t > 0$:

$$t \Psi_1(u) = \Psi_t(u).$$
For every Lévy process, the following property holds:

\[
E[e^{iuL_t}] = e^{t\Psi_1(u)}
\]

\[
= \exp[t(ibu - \frac{u^2c}{2}) + \int_{\mathbb{R}\backslash\{0\}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|<1\}}\nu(dx))]\]

where \( \Psi(u) = \Psi_1(u) \) is the characteristic exponent of \( L_1 = X \).
Lévy Processes

The triplet \((b, c, \nu)\) is called the Lévy or characteristic triplet and

\[
\Psi(u) = i bu - \frac{u^2 c}{2} + \int_{\mathbb{R}\setminus \{0\}} (e^{iux} - 1 - iux1_{\{|x|<1\}}) \nu(dx)
\]

is called the Lévy or characteristic exponent,

where \(b \in \mathbb{R}\) : drift term,

\(c \in \mathbb{R}^+\) : diffusion coefficient and

\(\nu\) : Lévy measure.
Another key concept, the *Lévy-Ito Decomposition Theorem*, allows one to describe the structure of a Lévy process sample path.

**Lévy-Ito Decomposition Theorem:**

Let \( X \) be a Lévy process, the distribution of \( X_{1} \) parametrized by \((\beta, \sigma^{2}, \nu)\). Then \( X \) decomposes as \( X_{t} = \beta t + \sigma W_{t} + J_{t} + M_{t} \), where \( W_{t} \) is a Brownian motion, and \( \Delta X_{t} = X_{t} - X_{t^{-}} \) (\( t \geq 0 \)) is an independent Poisson point process with intensity measure \( \nu \), \( J_{t} = \sum_{s \leq t} \Delta X_{s} 1_{\{\Delta X_{s} \geq 1\}} \) and \( M_{t} \) is a martingale with jumps, \( \Delta M_{t} = \Delta X_{t} 1_{\{\Delta X_{t} < 1\}} \).
Basic Idea

\[ X(k + 1) = \mathbf{M}_{s(k)} X(k) + \mathbf{C}_{s(k)} \]

\[ s(k) := F_B Q(X(k - 1)) \]

\[ Q_i(X(k)) := \begin{cases} 
1 & \text{if } X_i(k) > \theta_i \\
0 & \text{otherwise} 
\end{cases} \]
∃ \mathcal{O}_{(f,h,g,u,v)} \forall (\tilde{f}, \tilde{h}, \tilde{g}, \tilde{u}, \tilde{v}) \in \mathcal{O}_{(f,h,g,u,v)} \exists \psi(\cdot), \varphi(\cdot, \cdot) \in C^0:\n
\begin{align*}
M_{\mathcal{G}_{SI}}[h, g, u, v] & \quad \varphi(\cdot, \tau) \\
M_{\mathcal{G}_{SI}}[\tilde{h}, \tilde{g}, \tilde{u}, \tilde{v}] & \quad \text{homeom.}
\end{align*}
constant \((n \times n)\)-matrix with entries \(m_{ij}\) representing the effect which the expression level of gene \(j\) has on the change of expression of gene \(i\) genetic regulation network

**mixed-integer nonlinear optimization problem (MINLP):**

\[
\min_{M=(m_{ij})} \sum_{k=1}^{l} \left\| M E^{(k)} - \dot{E}^{(k)} \right\|^2_2, \\
i, j \in G, G = \{1, \ldots, n\}
\]

subject to

\[
\left\{ \begin{array}{l}
m_{ij} \geq \begin{cases} -\lambda(i), & i = j, \\
0, & i \neq j. \end{cases} \\
(1 - y_{ij}) \cdot m_{ij} = 0, & \forall i, j \in G. \\
\sum_{j \in G} y_{ij} \leq \deg_{\text{max},i}, & \forall i \in G.
\end{array} \right.
\]

Binary variables \(y_{ij} \in \{0, 1\}:

\[
y_{ij} = \begin{cases} 0, & \text{if } m_{ij} = 0, \\
1, & \text{if } m_{ij} \neq 0. \end{cases}
\]
Results of 3rd-order Heun Method for all genes:

- gene A
- gene B
- gene C
- gene D
Stochastic Hybrid Model I

\[ (X(t), \theta(t))_{t \in [0, \infty)} : \]

\[ X(t) \in \mathbb{R}^d, \quad \theta(t) \in \Theta = \{0, 1, \ldots, N\}, \quad \text{where} \]

- \( \theta(t) \) is a pure jump process taking values in \( \Theta \),
- \( (X(t), \theta(t)) \) is a switching jump-diffusion between jumps.

M.K. Ghosh and A. Bagchi, 2004
The asset price dynamics is given by the following system:

\[
\begin{aligned}
    dX(t) &= \mu(X(t), \theta(t))dt + \sigma(X(t), \theta(t))dW(t) + \int_{\mathbb{R}} g(X(t^-), \theta(t^-), u)N(dt, du), \\
    d\theta(t) &= \int_{\mathbb{R}} h(X(t^-), \theta(t^-), u)N(dt, du), \\
    X(0) &= X_0, \quad \theta(0) = \theta_0.
\end{aligned}
\]

for \( t \geq 0 \),

The price process \( X(t) \) switches from one jump-diffusion path to another as the discrete component \( \theta(t) \) moves from one step to another.
Stochastic Control of Hybrid Systems

\[ h : \mathbb{R}^d \times \Theta \times \mathbb{R} \rightarrow \mathbb{R}^N \] is defined as

\[ h(x, i, u) = \begin{cases} j - i, & \text{if } u \in \Delta_{ij}(x), \\ 0, & \text{otherwise,} \end{cases} \]

\( N(\cdot) \): Poisson random measure,
\( \tilde{N}(\cdot) \): Poisson process on \((\Omega, \mathcal{F}, P)\) corresponding to the given Poisson random measure,
\( D_{\tilde{N}} \): domain of \( \tilde{N}(\cdot) \),
\[ D = \left\{ t \in D_{\tilde{N}} \mid \tilde{N}(t) \in U \right\}, \]
\[ \mathcal{F}_t = \sigma\{W(s), N(A, B) \mid s \leq t, A \in \mathcal{B}([0, T]), B \in \mathcal{B}(\mathbb{R})\} \]
Merton’s Consumption Investment Problem

An investor with a finite lifetime must determine the amounts \( c_t \) that will be consumed and the fraction of wealth \( \pi_t \) that will be invested in a stock portfolio, so as to maximize expected lifetime utility.

Assuming a relative consumption rate

\[
\lambda(t) = \frac{c(t)}{X^{(c,\pi)}(t)},
\]

the wealth process evolves with the following SDE:

\[
d^{-} X^{(\lambda,\pi)}(t, \theta_t) = X^{(\lambda,\pi)}(t, \theta_t) \left\{ \left[ r(t, \theta_t) - \lambda(t, \theta_t) \right] + (\mu(X^{(\lambda,\pi)}_t, \theta_t) - r(t, \theta_t))\pi(t, \theta_t) \right\} d^{-} t \\
+ \sigma(X^{(\lambda,\pi)}_t, \theta_t)\pi(t, \theta_t)d^{-} B(t) + \pi(t, \theta_t) \int_{\mathbb{R}_0} g(X^{(\lambda,\pi)}_{t^{-}}, \theta_{t^{-}}, u)N(dt^{-}, du). \]

D. David and Y. Yolcu Okur, 2009
Merton’s Consumption Investment Problem

We assume a *constant relative risk aversion* (CRRA) utility function

\[ U(x, t) = \frac{x^{1-\gamma}}{1-\gamma}, \]

where \( \gamma \in (0, \infty) \setminus \{1\}. \)

Hence, the objective takes the following form

\[
\max_{(c, \pi)} E\left[ \int_{0}^{T} e^{-\delta s} U(c(s), s) ds + e^{-\delta T} U(X^{(c, \pi)}(T), T) \right]
\]

subject to \( c(t) \geq 0, \ X^{(c, \pi)}(t) > 0 \ (t \in [0, T]), \ X^{(c, \pi)}(0) = \nu. \)
Hamilton-Jacobi-Bellman Equation

In applying the *dynamic programming* approach, we solve the Hamilton-Jacobi-Bellman (HJB) equation associated with the utility maximization problem (2).

From *(W. Fleming and R. Rishel, 1975)* we have that the corresponding **HJB equation** is given by

\[
\rho J (\nu, t) = \max_{c_t \geq 0, \pi_t \in \mathbb{R}^n} \left\{ U (c_t) + J_t (\nu, t) + J_v (\nu, t) \left( \nu \left[ \pi_T^T (\mu - r) + r \right] - c_t \right) + \frac{1}{2} J_{vv} (\nu, t) \nu^2 \pi_T^T \pi \right\}
\]

subject to the terminal condition \( J (\nu, T) = u (\nu) \),

where \( J \) the **value function** is given by

\[
J (\nu, t) = \sup_{(c, \pi)} \left[ E \left[ \int_t^T e^{-\delta s} U (c_s, s) \, ds + e^{-\delta T} U (X_{c, \pi} (T), T) \right] \right].
\]
Hamilton-Jacobi-Bellman Equation

In solving the HJB equation (3), the static optimization problem

$$\max_{c_t \geq 0, \pi_t \in \mathbb{R}^n} \left\{ U(c_t) + J_v(v,t)\left( v\left[ \pi_t^T (\mu - r) + r \right] - c_t \right) + \frac{1}{2} J_{vv}(v,t) v^2 \pi_t^T \sigma \sigma^T \pi_t \right\}$$

can be handled separately to reduce the HJB equation (3) to a nonlinear partial differential equation of $J$.

Introducing the Lagrange function as

$$L(\pi(v,t), c(v,t), \tilde{\lambda}(v,t)) = J_v(v,t) \left( v\left[ \pi_t^T (\mu - r) + r - c_t \right] \right) + \frac{1}{2} v^2 \left\| \pi_t^T \sigma \right\|^2 J_{vv}(v,t) + u(c_t) - \tilde{\lambda}(v,t) c_t, \quad (4)$$

where $\tilde{\lambda}$ is the Lagrange multiplier.
Simultaneous resolution of these first-order conditions yields the optimal solutions $\pi^{opt}, c^{opt}$ and $\tilde{\lambda}^{opt}$.

Substituting these into (3) gives the PDE

$$-\delta J(v,t) + \left(\frac{c^{opt}(v,t)}{1-\gamma}\right) + J_t(v,t) + J_v(v,t)\left(v\left[(\pi^{opt}(v,t))^T(\mu-r) + r\right] - c^{opt}(v,t)\right)$$

$$+ \frac{1}{2} J_{vv}(v,t)v^2\left(\pi^{opt}(v,t))^T\sigma\sigma^T(\pi^{opt}(v,t)) = 0,$$

with terminal condition

$$J(v,T) = e^{-\delta T} \frac{v^{1-\gamma}}{1-\gamma},$$

which can then be solved for the optimal value function $J^{opt}(v,t)$. 
Because of the nonlinearity in $\pi^{opt}$ and $c^{opt}$, the first-order conditions together with the HJB equation are a nonlinear system.

So (3) and (5) do not always have an analytic solution and numerical methods such as Newton’s method or Sequential Quadratic Programming (SQP) are required to solve for $\pi^{opt}(\nu,t)$, $c^{opt}(\nu,t)$, $\tilde{\lambda}^{opt}(\nu,t)$ and $J^{opt}(\nu,t)$ iteratively.
Stochastic Control of Hybrid Systems

Stochastic Hybrid Model II

\[ dX(t) = b^n(X(t), \theta(t))dt + \sigma^n(X(t), \theta(t))dW^n(t), \]

\[ P(\theta(t + \delta t) = j \mid \theta(t) = i, X(s), \theta(s), s \leq t) = \lambda_{ij}^n(X(t))\delta t + O(\delta t) \quad \forall i \neq j, \]

\[ X(0) = x_0, \quad \theta(0) = \theta_0 \]
Stochastic Hybrid Model II

\[ dX(t) = b^n(X(t), \theta(t))dt + \sigma^n(X(t), \theta(t))dW^n(t), \]

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\[ X(0) = x_0, \quad \theta(0) = \theta_0 \]
Stochastic Hybrid Model II

\[ \mathcal{F}_t^n = \sigma \{ W^n(s), N(A, B) \mid 0 \leq s \leq t, \ A \in \mathcal{B}([0, t]), B \in \mathcal{B}(\mathbb{R}) \} , \]

\[ \tau_n = \inf \{ t \geq 0 \mid X(t) \in A_n \} , \]

\[ \mathcal{F}_t = \bigvee_{n \in \mathbb{N}} \mathcal{F}_t^n , \]

\[ 0 = \tau_0 < \tau_1 < \ldots < \tau_m < \ldots , \quad \tau_m \to \infty \ (m \to \infty) , \]

\[ \eta(t) = n , \quad \text{if} \ (X(t), \theta(t)) \in S_n \times \Theta_n . \]
Stochastic Hybrid Model II

\[ dX(t) = \left( b(X(t), \theta(t), \eta(t)) + \sum_{m=0}^{\infty} \left( \tilde{g}_1(X(\tau_m^-), \theta(\tau_m^-), \eta(\tau_m^-)) - X(\tau_m^-) \right) \delta(t - \tau_m) \right) dt + \sigma(X(t), \theta(t), \eta(t)) dW^{\eta(t)}(t), \]

\[ d\theta(t) = \int_{\mathbb{R}} h(X(t^-), \theta(t^-), \eta(t^-), u) N(dt, du) + \sum_{m=0}^{\infty} \left( \tilde{g}_2(X(\tau_m^-), \theta(\tau_m^-), \eta(\tau_m^-)) - \theta(\tau_m^-) \right) \delta(t - \tau_m) dt, \]

\[ d\eta(t) = \sum_{m=0}^{\infty} \left( \tilde{h}(X(\tau_m^-), \theta(\tau_m^-), \eta(\tau_m^-)) - \eta(\tau_m^-) \right) I_{\{\tau_m \leq t\}} \]
Stochastic Control of SHS with Jumps

(a) Discrete Approximation: B.-Z. Temocin, G.-W. Weber

(b) Closed-Form Solutions: N. Azevedo, D. Pinheiro, G.-W. Weber
(a) **Discrete Approximation:** B.-Z. Temocin, G.-W. Weber

(b) **Closed-Form Solutions:** N. Azevedo, D. Pinheiro, G.-W. Weber

**Dynamic programming for a Markov-switching jump-diffusion**

JCAM 267: 1–19 (2014)
Merton’s Consumption Investment Problem

Setup:
- Standard one-dimensional Brownian motion $W = (W(t) : t \in [0, T])$.
- One-dimensional Lévy process $(\eta(t) : t \in [0, T])$ with Poisson random measure $J(t, A)$ with finite intensity $\nu(A) = E[J(1, A)]$.
- Continuous time Markov process $(\alpha(t) : t \in [0, T])$ with a finite space state $S = \{a_1, \ldots, a_n\}$ and generator $Q = (q_{ij})_{i,j \in S}$.
- $W(\cdot), \eta(\cdot), \alpha(\cdot)$ independent.

Risk-free asset:

$$dS_0(t) = r(t, \alpha(t_-))S_0(t_-)dt, \quad S_0(0) = s_0 > 0.$$ 

Risky asset:

$$dS_1(t) = \mu(t, \alpha(t_-))S_1(t_-)dt + \sigma(t, \alpha(t_-))S_1(t_-)dW(t) + S_1(t_-) \int_{\mathbb{R}_0^1} h(t, \alpha(t_-), z) \tilde{J}(dt, dz), \quad S_1(0) = s_1 > 0.$$
Merton’s Consumption Investment Problem

Control variables

- *consumption process* \{c(t) : t \in [0, T]\}.
- *portfolio process* \{\Theta(t) : t \in [0, T]\}

\[ \Theta(t) = (\theta_0(t), \theta_1(t)) \in [0, 1] \times [0, 1], \]

where

\[ \theta_0(t) + \theta_1(t) = 1, \quad 0 \leq t \leq T. \]

State variable

- *wealth process* \{X(t) : t \in [0, T]\}

\[ dX(t) = \left( -c(t) + \left( \theta_0(t)r(t, \alpha(t_-)) + \theta_1(t)\mu(t, \alpha(t_-)) \right) X(t_-) \right) dt \]

\[ + \theta_1(t)X(t_-) \left( \sigma(t, \alpha(t_-))dW(t) + \int_{\mathbb{R}_{t_0}^{1}} \eta(t, \alpha(t_-), z)\tilde{J}(dt, dz) \right). \]
Power Utility Functions

We assume that the utility functions belong to the class of constant relative risk aversion (CRRA).

Let

\[ U(t, c, a) = e^{-\rho t} \frac{c^{\gamma_a}}{\gamma_a}, \quad \Psi(t, x, a) = e^{-\rho t} \frac{x^{\gamma_a}}{\gamma_a} \]

where

\[ \gamma_a \in (0, 1) \] is the risk aversion coefficient associated with the state of the Markov process \( \alpha(t) = a \in S; \)

\[ \rho > 0 \] is the discount rate.

Let \( F : [0, 1] \times [0, T] \times S \rightarrow \mathbb{R} \) be given by

\[
F(\theta; t, a) = \gamma_a \left[ r(t, a) + \theta(\mu(t, a) - r(t, a)) - \frac{1}{2} (1 - \gamma_a) \theta^2 \sigma^2(t, a) \right] \\
+ \int_{\mathbb{R}_0^1} \left( (1 + \theta \eta(t, a, z))^{\gamma_a} - 1 - \gamma_a \theta \eta(t, a, z) \right) \nu(dz).
\]
Theorem (Power utility functions)

The maximum expected utility is given by

$$V(t, x, a) = \xi_a(t)^X_{1}^{\gamma_{a} \gamma_{a}}$$

and the corresponding optimal strategies are of the form

$$c^*(t, x, a) = x(e^{\rho t} \xi_a(t))^{-1/(1-\gamma a)}$$

and

$$\theta^*(t, a) = \begin{cases} 1 & \text{if } \mu(t, a) > r(t, a) \quad \text{and} \quad F'(1; t, a) \geq 0 \\ \hat{\theta}(t, a) & \text{if } \mu(t, a) > r(t, a) \quad \text{and} \quad F'(1; t, a) < 0 \\ 0 & \text{if } \mu(t, a) \leq r(t, a) \end{cases}$$

where \( \hat{\theta}(t, a) \) is the unique solution of \( F'(\theta; t, a) = 0 \) in \((0, 1)\) and \( \xi_a(t), a \in S, \) are the solutions of the following coupled ordinary differential equations terminal value problem

$$\xi'_a(t) + (1 - \gamma a)e^{-\rho t/(1-\gamma a)}\xi_a(t)^{\gamma a/(1-\gamma a)} + F(\theta^*(t, a); t, a)\xi_a(t)$$

$$+ \sum_{j \in S: j \neq a} q_{aj}(\xi_j(t) - \xi_a(t)) = 0$$

$$\xi_a(T) = e^{-\rho T}.$$
A simple example

Consider a finite space state $S = \{+, 0, -\}$.

A reasonable choice of parameters should satisfy:

1) $r_+ > r_0 > r_- \geq 0$;
2) $\mu_+ > \mu_0 > \mu_-;
3) $\mu_+ > r_+, \quad \mu_0 \approx 0, \quad \mu_- < 0$;
4) $\sigma_+ < \sigma_0 < \sigma_-.$

one risk-free security and one risky asset driven by a jump diffusion market with regime-switching.
A simple example

We made the following choice of parameters:

- \( T = 20 \).
  - (i) State \(+\): \( r_+ = 0.05, \mu_+ = 0.20 \) and \( \sigma_+ = 0.11 \).
  - (ii) State \( 0 \): \( r_0 = 0.03, \mu_0 = 0.04 \) and \( \sigma_0 = 0.21 \).
  - (iii) State \(-\): \( r_- = 0.01, \mu_- = -0.1 \) and \( \sigma_- = 0.40 \).

- In what concerns the jump process, we take \( h(t, a, z) = z \). Let
  - \( \nu(dz) \) be the measure associated with a uniform distribution with support \([-0.25, 0.25] \);
  - the waiting time between jumps be exponentially distributed with mean \( \lambda = 1 \).

- The generator of the Markov process \( \alpha(\cdot), Q = (q_{ij})_{i,j \in S} \), is determined by
  - \( q_{+0} = 3, q_{+\cdot} = 6, q_{0+} = 1, q_{0\cdot} = 1, q_{\cdot+} = 2 \) and \( q_{\cdot0} = 1 \).

- CRRA utilities with risk aversion parameters \( \gamma_+ = 0.6, \gamma_0 = 0.5 \) and \( \gamma_- = 0.4 \)

- \( \rho = 0.03 \).
A simple example

**Figure:** Sample path of the risk free asset price evolution with time.

**Figure:** Path for the risky asset.
A simple example

Figure: Sample path of the wealth process associated with the optimal portfolio for the consumption investment problem.
A simple example

Figure: Plot of proportion of wealth consumed. The $x$ axis represents the time, and the $y$ axis the consumption rate $c$. 
A simple example

Figure: Optimal consumption path for the consumption investment problem.
Stochastic Control of SHS with Jumps and Delay

(c) Analysis and Closed-Form Solutions: Emel Savku, G.-W. Weber

Maximum Principle for a Delayed Markov-switching jump-diffusion
We consider a stochastic optimal control problem where the system is given by a stochastic differential delay equation with jumps (SDDEJ) and Markov regime-switches.

The stochastic maximum principle states that any optimal control satisfies a system of forward-backward SDDEs, called the *optimality system*, and maximizes a functional, called the *Hamiltonian*.

The converse is also true, giving the sufficient maximum principle.
By regime-switches, we can capture:

- The changes in the behaviors of financial markets, e.g., a shift from a volatile period to a calm period.
- The movements of financial markets between different states, e.g., growth, recession, crisis and bubble.
- Discrete shifts from one regime to another may be activated by a change in economic policy, e.g., a shift in a monetary or an exchange rate policy or by a major event, e.g., the bankruptcy of Lehman Brothers in September 2008 or the 1973 oil crisis.
Delay – Maximum Principle for a Markov Regime-Switching Jump-Diffusion Model

- The delays in the dynamics can represent memory or inertia in the financial system.
- In the real world, investors tend to look at the historic performance of the risky asset.
  - If the price increases a lot, more investors tend to invest more money in this asset.
  - If the price decreases a lot, more investors tend to sell the asset and invest in other assets.
Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a complete filtered probability space where \(\mathbb{F} = \{\mathcal{F}(t) : t \in [0, T]\}\) is a right continuous, \(\mathbb{P}\)-completed filtration.

Let \(\{\alpha(t) : t \in [0, T]\}\) be a homogeneous, irreducible continuous-time observable Markov chain with finite state space \(S := \{e_1, e_2, \ldots, e_D\}\), where \(D \in \mathbb{N}\), \(e_i \in \mathbb{R}^D\).

Let the \(j\)th component of \(e_i\) be the Kronecker delta \(\delta_{ij}\) for each \(i, j = 1, 2, \ldots, D\).

Let \(\Lambda := [\lambda_{ij}]_{i,j=1,2,\ldots,D}\) be the generator of the chain under \(\mathbb{P}\), where for each \(i, j = 1, 2, \ldots, D\), \(\lambda_{ij}\) is the constant transition intensity of the chain from state \(e_i\) to state \(e_j\) at time \(t\).

Let \(\lambda_{ij} \geq 0\) for \(i \neq j\) and \(\sum_{j=1}^{D} \lambda_{ij} = 0\).

Let Brownian motion \(W(\cdot)\), the Markov chain \(\alpha(\cdot)\), Poisson random measure \(N(\cdot, \cdot)\) be independent and adapted to \(\mathbb{F}\).
Elliot, Aggoun and Moore (1994) obtained the following semimartingale dynamics for the chain $\alpha$:

$$\alpha(t) = \alpha(0) + \int_0^t \Lambda^T \alpha(u) du + M(t),$$

where $M(t)$ is a an $\mathbb{R}^D$-valued ($\mathbb{F}, \mathbb{P}$)-martingale. Let us define:

$$J^{ij}(t) := \sum_{0 < s \leq t} \langle \alpha(s^-), e_i \rangle \langle \alpha(s), e_j \rangle$$

$$= \lambda_{ij} \int_0^t \langle \alpha(s^-), e_i \rangle ds + \int_0^t \langle \alpha(s^-), e_i \rangle \langle dM(s), e_j \rangle.$$

We represent:

$$m_{ij}(t) := \int_0^t \langle \alpha(s^-), e_i \rangle \langle dM(s), e_j \rangle,$$

which is an ($\mathbb{F}, \mathbb{P}$)-martingale.
For each fixed $j = 1, 2, \ldots, D$, let $\Phi_j$ be the number of jumps into state $e_j$ up to time $t$. Then,

$$\Phi_j(t) := \sum_{i=1, i \neq j}^{D} J^{ij}(t)$$

$$= \sum_{i=1, i \neq j}^{D} \lambda_{ij} \int_{0}^{t} \langle \alpha(s-), e_i \rangle \, ds + \tilde{\Phi}_j(t),$$

where $\tilde{\Phi}_j(t) := \sum_{i=1, i \neq j}^{D} m_{ij}(t)$ and $\lambda_j := \sum_{i=1, i \neq j}^{D} \lambda_{ij} \int_{0}^{t} \langle \alpha(s-), e_i \rangle \, ds$.

Hence, $\tilde{\Phi}_j(t) = \Phi_j(t) - \lambda_j(t)$ is an $(\mathcal{F}, \mathbb{P})$-martingale.
Let us represent our model:

\[
\begin{aligned}
    dX(t) &= b(t, X(t), Y(t), A(t), \alpha(t), u(t))dt \\
    &\quad + \sigma(t, X(t), Y(t), A(t), \alpha(t), u(t))dW(t) \\
    &\quad + \int_{\mathbb{R}_0} \eta(t, X(t), Y(t), A(t), \alpha(t), u(t), z)\tilde{N}(dt, dz) \\
    &\quad + \gamma(t, X(t), Y(t), A(t), \alpha(t), u(t))d\tilde{\Phi}(t), \quad t \in [0, T] \\
    X(t) &= x_0(t), \quad t \in [-\delta, 0] \text{ and } x_0 \in C([-\delta, 0]; \mathbb{R}),
\end{aligned}
\]  

where

\[Y(t) = X(t - \delta) \quad \text{and} \quad A(t) = \int_{t-\delta}^{t} e^{\rho(t-r)}X(r)dr \quad \text{for} \quad t \in [0, T].\]

\(\delta > 0, \rho \geq 0\) and \(T > 0\) are given constants and \(b, \sigma, \eta\) and \(\gamma\) are \(\mathcal{F}_t\)-measurable functions for all \(t\).

Here, \(\tilde{\Phi}(t) := (\tilde{\Phi}_1(t), \tilde{\Phi}_2(t), \ldots, \tilde{\Phi}_D(t))^T\) be the Markov jump-martingales associated with the chain \(\alpha\).
Let $U$ be a non-empty, closed, convex subset of $\mathbb{R}$. An admissible control is a $U$-valued, $\mathcal{F}_t$-measurable, càdlàg process $u(t), t \in [0, T]$, such that the SDDE (1) has a unique solution and

$$E \left[ \int_0^T |u(t)|^2 dt \right] < \infty.$$

We denote by $\mathcal{A}$ the set of all admissible controls. Let us define the performance criterion (objective functional) as follows:

$$J(u) = E \left[ \int_0^T f(t, X(t), Y(t), A(t), \alpha(t), u(t)) dt + g(X(T), \alpha(T)) \right]$$

for all $u \in \mathcal{A}$, where $f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times S \times U \to \mathbb{R}$ and $g : \mathbb{R} \times S \to \mathbb{R}$ are $C^1$-functions with respect to $x, y, a, u$ such that $x_i = x, y, a, u,$
Delay – Markov-Switching Jump-Diffusion

\[
E \left[ \int_0^T \left( \left| f(t, X(t), Y(t), A(t), \alpha(t), u(t)) \right| + \left| \frac{\partial f(t, X(t), Y(t), A(t), \alpha(t), u(t))}{\partial x_i} \right|^2 \right) \, dt + \left| g(X(T), \alpha(T)) \right| + \left| g_x(X(T), \alpha(T)) \right|^2 \right] < \infty.
\]

Our problem is to find an optimal control \( \hat{u} \in \mathcal{A} \) such that

\[
J(\hat{u}) = \sup_{u \in \mathcal{A}} J(u(\cdot)) = \sup_{u \in \mathcal{A}} E \left[ \int_0^T f(t, X(t), Y(t), A(t), \alpha(t), u(t)) \, dt + g(X(T), \alpha(T)) \right].
\]
We define the Hamiltonian,

\[ H(t, x, y, a, e_i, u, p, q, r, s) = f(t, x, y, a, e_i, u) + b(t, x, y, a, e_i, u)p + \sigma(t, x, y, a, e_i, u)q + \int_{\mathbb{R}_0} \eta(t, x, y, a, e_i, u, z)r(t, z)\nu(dz) + \sum_{j=1}^{D} \gamma^j(t, x, y, a, e_i, u)\omega^j(t)\lambda_{ij}. \]

The adjoint equations corresponding to \( u \) and \( X(t) := X^{(u)}(t) \) for the unknown, adapted processes \( \{p(t) | t \in [0, T]\} \), \( \{q(t) | t \in [0, T]\} \), \( \{r(t, z) | t \in [0, T], z \in \mathbb{R}_0\} \), and \( \{\omega(t) | t \in [0, T]\} \), where \( p(t) \in \mathbb{R} \), \( q(t) \in \mathbb{R} \), \( r(t, z) \in \mathbb{R} \), and \( \omega(t) \in \mathbb{R}^D \) are given by the following Anticipated Backward Stochastic Differential Equation (ABSDE):
\[
\begin{cases}
    dp(t) = E[\mu(t)|\mathcal{F}_t]dt + q(t)dW(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz) + \omega(t)d\tilde{\Phi}(t), \\
p(T) = g_x(X(T),\alpha(T)),
\end{cases}
\]

where

\[
\begin{cases}
    \mu(t) = -\frac{\partial H}{\partial x}(t,X(t),Y(t),A(t),\alpha(t),u(t),p(t),q(t),r(t,\cdot),\omega(t)) \\
    -\frac{\partial H}{\partial y}(t+\delta,X(t+\delta),Y(t+\delta),A(t+\delta),\alpha(t+\delta), \\
u(t+\delta),p(t+\delta),q(t+\delta),r(t+\delta,\cdot),\omega(t+\delta))\chi_{[0,T-\delta]}(t) \\
e^{\rho t}(\int_{t}^{t+\delta} \frac{\partial H}{\partial a}(s,X(s),Y(s),A(s),\alpha(s),u(s),p(s),q(s), \\
r(s,\cdot),\omega(s))e^{-\rho s}\chi_{[0,T]}(s)ds).
\end{cases}
\]
We will consider a new form for the BSDEs as follows:

\[-dY(t) = f(t, Y(t), Z(t), Q(t), V(t), Y(t + \delta_1(t)), Z(t + \delta_2(t)), Q(t + \delta_3(t)), V(t + \delta_4(t)))dt - Z(t)dW(t) - \int_{\mathbb{R}_0} Q(t, z)\tilde{N}(dt, dz) - V(t)d\tilde{\Phi}(t), \quad t \in [0, T];\]

\[Y(t) = \xi(t), \quad Z(t) = \psi(t), \quad Q(t) = \zeta(t) \text{ and } V(t) = \vartheta(t), \quad t \in [T, T + K].\]

Let $\delta_i(\cdot), \ i = 1, 2, 3, 4,$ be $\mathbb{R}^+$-valued continuous functions on $[0, T]$ such that:

**(i)** There exists a constant $K \geq 0$ such that for all $s \in [0, T]$ and $i = 1, 2, 3, 4$, $s + \delta_i(s) \leq T + K$.

**(ii)** There exists a constant $L \geq 0$ such that for all $t \in [0, T]$ and some non-negative integrable function $g(\cdot)$,

\[\int_t^T g(s + \delta_i(s))ds \leq L \int_t^{T+K} g(s)ds \quad \text{for} \quad i = 1, 2, 3, 4.\]
Let us assume:

(H1) There exists a constant $C > 0$ such that for all $s \in [0, T], y, y', z, z' \in \mathbb{R}, q, q' \in L^2(\mathcal{B}_0; \mathbb{R}), v, v' \in L^2(\mathcal{B}_s; \mathbb{R}^D), \xi, \xi', \psi, \psi' \in L^2_F(s, T + K; \mathbb{R}), \zeta, \zeta' \in \mathcal{H}^2_F(s, T + K; \mathbb{R}), \vartheta, \vartheta' \in \mathcal{M}^2_F(s, T + K; \mathbb{R}^D)$ and $r, r^*, \hat{r}, \bar{r} \in [s, T + K]$, we have

$$
|f(s, y, z, q, v, \xi(r), \psi(r^*), \zeta(\hat{r}), \vartheta(\bar{r})) - f(s, y', z', q', v', \xi'(r), \psi'(r^*), \zeta'(\hat{r}), \vartheta'(\bar{r}))| \\
\leq C \left( |y - y'| + |z - z'| + \|q - q'\|_J + \|v - v'\|_S + E_{\mathcal{R}_s} \left[ |\xi(r) - \xi'(r)| \\
+ |\psi(r^*) - \psi'(r^*)| + \|\zeta(\hat{r}) - \zeta'(\hat{r})\|_J + \|\vartheta(\bar{r}) - \vartheta'(\bar{r})\|_S \right] \right).
$$

(H2) $E \left[ \int_0^T |f(s, 0, 0, 0, 0, 0, 0, 0)|^2 dt \right] < \infty.$
**Theorem:** Suppose \( f \) satisfies (H1) and (H2) and for \( i = 1, 2, 3, 4 \), \( \delta_i \) satisfies (i) and (ii). Then for any given terminal condition \( \xi(\cdot) \in S_F^2(T, T + K; \mathbb{R}) \), \( \psi(\cdot) \in L_F^2(T, T + K; \mathbb{R}) \), \( \zeta(\cdot) \in H_F^2(T, T + K; \mathbb{R}) \) and \( \vartheta(\cdot) \in M_F^2(T, T + K; \mathbb{R}^D) \), the ABSDE has a unique solution, i.e., there exists a unique 4-tuple of \( \mathcal{F}_t \)-adapted processes \( (Y, Z, Q, V) \in S_F^2(0, T + K; \mathbb{R}) \times L_F^2(0, T + K; \mathbb{R}) \times H_F^2(0, T + K; \mathbb{R}) \times M_F^2(0, T + K; \mathbb{R}^D) \) satisfying the ABSDE.
Delay – **Necessary Condition for Optimality**

Let \( \hat{u} \) be an optimal control process and \( \beta \) “small” be such that \( \hat{u} + \beta = v \in \mathcal{A} \). Since \( \mathcal{U} \) is a convex set, for any \( v \in \mathcal{A} \) nearby the perturbed control process \( u^s = \hat{u} + s(v - \hat{u}), \ 0 < s < 1 \), is also in \( \mathcal{A} \).

The directional derivative of the performance criterion \( J(\cdot) \) at \( \hat{u} \) in the direction of \( v - \hat{u} \) is given by:

\[
\frac{d}{ds} J(\hat{u} + s(v - \hat{u}))|_{s=0} := \lim_{s \to 0^+} \frac{J(\hat{u} + s(v - \hat{u})) - J(\hat{u})}{s}.
\]

Since \( \hat{u} \) is an optimal control, then a necessary condition for optimality is:

\[
\frac{d}{ds} J(\hat{u} + s(v - \hat{u}))|_{s=0} \leq 0.
\]

Let the derivative process \( \xi(t) := \frac{d}{ds}X^u + s\beta(t)|_{s=0} \) exist.
**Theorem:** Let \( \hat{u} \in A \) be an optimal control with the corresponding trajectory \((\hat{X}(t), \hat{Y}(t), \hat{A}(t))\) and \((\hat{p}(t), \hat{q}(t), \hat{r}(t, z), \hat{w}(t))\) be the unique solution of the corresponding adjoint equation. Assume that,

\[
E \left[ \int_0^T \hat{p}^2(t) \left\{ \left( \frac{\partial \hat{\sigma}}{\partial x} \right)^2(t) \hat{\xi}^2(t) + \left( \frac{\partial \hat{\sigma}}{\partial y} \right)^2(t) \hat{\xi}^2(t-\delta) + \left( \frac{\partial \hat{\sigma}}{\partial u} \right)^2(t) \right\} + \left( \frac{\partial \hat{n}}{\partial u} \right)^2(t) \beta^2(t) + \int_{\mathbb{R}_0} \left\{ \left( \frac{\partial \hat{n}}{\partial x} \right)^2(t, z) \hat{\xi}^2(t) \right. \\
\left. + \left( \frac{\partial \hat{n}}{\partial y} \right)^2(t, z) \hat{\xi}^2(t-\delta) + \left( \frac{\partial \hat{n}}{\partial a} \right)^2(t, z) \left( \int_{t-\delta}^t e^{\rho(t-r)} \hat{\xi}(r) dr \right)^2 + \left( \frac{\partial \hat{n}}{\partial u} \right)^2(t, z) \beta^2(t) \right\} v(dz) + \sum_{j=1}^D \left\{ \left( \frac{\partial \hat{\gamma}^j}{\partial x} \right)^2(t) \hat{\xi}^2(t) + \left( \frac{\partial \hat{\gamma}^j}{\partial y} \right)^2(t) \hat{\xi}^2(t-\delta) \right. \\
\left. + \left( \frac{\partial \hat{\gamma}^j}{\partial a} \right)^2(t) \left( \int_{t-\delta}^t e^{\rho(t-r)} \hat{\xi}(r) dr \right)^2 + \left( \frac{\partial \hat{\gamma}^j}{\partial u} \right)^2(t) \beta^2(t) \right\} \lambda_j(t) \right\} dt \right] < \infty
\]
Delay – **Necessary Maximum Principle**

and

\[
E \left[ \int_0^T (\xi'(t))^2(t) \left\{ (\dot{q})^2(t) + \int_{R_0^0} (\dot{r})^2(t, z) \nu(dz) + \sum_{j=1}^{D} (\dot{\omega}^j)^2(t) \lambda_j(t) \right\} dt \right] < \infty.
\]

Then, for any \( v \in U \) we have

\[
\frac{\partial H}{\partial u}(t, \dot{X}(t), \dot{Y}(t), \dot{A}(t), \alpha(t), \dot{u}(t), \dot{p}(t), \dot{q}(t), \dot{r}(t, \cdot), \dot{\omega}(t))(v - \dot{u}(t)) \leq 0
\]

\( dt - a.e, \quad \mathbb{P} - a.s. \).
Delay – **Sufficient Maximum Principle**

**Theorem:** Let \( \hat{u} \in \mathcal{A} \) with corresponding state processes \( \hat{X}(t), \hat{Y}(t), \hat{A}(t) \) and adjoint processes \( \hat{p}(t), \hat{q}(t), \hat{r}(t, z), \hat{\omega}(t) \), assumed to satisfy the ABSDE (2)-(3). Suppose that the following assertions hold:

\[
E \left[ \int_0^T \hat{p}(t)^2 ((\sigma(t) - \hat{\sigma}(t))^2 + \int_{\mathbb{R}_0} (\eta(t, z) - \hat{\eta}(t, z))^2 \nu(dz) \\
+ \sum_{j=1}^D (\gamma^j(t) - \hat{\gamma}^j(t))^2 \lambda_j(t)) \, dt \right] < \infty
\]

and

\[
E \left[ \int_0^T (X(t) - \hat{X}(t))^2 \left\{ \hat{\gamma}^2(t) + \int_{\mathbb{R}_0} \hat{r}^2(t, z) \nu(dz) + \sum_{j=1}^D (\hat{\omega}^j)^2(t) \lambda_j(t) \right\} \, dt \right] < \infty
\]
Furthermore, we assume that the following conditions hold:

1. For almost all \( t \in [0, T] \),

\[
H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \alpha(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{\omega}(t))
\]

\[
= \max_{u \in U} H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \alpha(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{\omega}(t)).
\]

2. \( H \) is a concave function of \( x, y, a, u \) for all \( (t, e_i) \in [0, T] \times S \).

3. \( g(x, e_i) \) is a concave function of \( x \) for each \( e_i \in S \).

Then \( \hat{u}(t) \) is an optimal control process and \( \hat{X}(t), \hat{Y}(t), \hat{A}(t) \) are the corresponding controlled state processes.
Delay – Partial Information

Let $\mathcal{E}_t \subseteq \mathcal{F}_t$, $t \in [0, T]$, be a given subfiltration of $\{\mathcal{F}_t\}_{t \in [0, T]}$ that represents the information available to the controller who decides the value of $u(t)$ at time $t$. For example, $\mathcal{E}_t = \mathcal{F}_{(t-c)^+}$ for some given $c > 0$.

Let $U \subset \mathbb{R}$ be a given set of admissible control values $u(t)$, $t \in [0, T]$, and $\mathcal{A}_\mathcal{E}$ be a given family of admissible control processes included in the set of càdlàg, $\mathcal{E}$-adapted, $U$-valued processes such that the defined SHDDEJ, equation (1), has unique solution.
Delay – Partial Information

Let $\mathcal{F}_t \subseteq \mathcal{F}_t$, $t \in [0, T]$, be a given subfiltration of $\{\mathcal{F}_t\}_{t \in [0, T]}$ that represents the information available to the controller who decides the value of $u(t)$ at time $t$. For example, $\mathcal{F}_t = \mathcal{F}_{(t-c)+}$ for some given $c > 0$.

Let $U \subset \mathbb{R}$ be a given set of admissible control values $u(t)$, $t \in [0, T]$, and $\mathcal{A}_\mathcal{F}$ be a given family of admissible control processes included in the set of càdlàg, $\mathcal{F}$-adapted, $U$-valued processes such that the defined SHDDEJ, equation (1), has unique solution.

Appendix
We represent the model for an application to optimal consumption from a cash flow with delay for a Markov regime-switching jump-diffusion market:

- Let \( b(t, \alpha(t)) \), \( \sigma(t, \alpha(t)) \), \( \eta(t, \alpha(t)) \) and \( \gamma(t, \alpha(t)) \) be given bounded adapted processes.
- Let the consumption rate at any time \( t \in [0, T] \), \( c(t) \geq 0 \), be a càdlàg adapted process.
- Then the corresponding net cash flow \( X(t) = X^c(t) \) is:

\[
\begin{align*}
  dX(t) &= (Y(t)b(t, \alpha(t)) - c(t))dt + Y(t)\sigma(t, \alpha(t))dW(t) \\
  &+ Y(t) \int_{\mathbb{R}_0} \eta(t, \alpha(t), z)\tilde{N}(dt, dz) \\
  &+ Y(t)\gamma(t, \alpha(t))d\tilde{\Phi}(t), \quad t \in [0, T]; \\
  X(t) &= x_0(t), \quad t \in [-\delta, 0] \text{ and } x_0 \in C([-\delta, 0]; \mathbb{R}).
\end{align*}
\]

(4)
We represent the model for an application to optimal consumption from a cash flow with delay for a Markov regime-switching jump-diffusion market:

- Let $b(t, \alpha(t))$, $\sigma(t, \alpha(t))$, $\eta(t, \alpha(t))$ and $\gamma(t, \alpha(t))$ be given bounded adapted processes.

- Let the consumption rate at any time $t \in [0, T]$, $c(t) \geq 0$, be a càdlàg adapted process.

- Then the corresponding net cash flow $X(t) = X^c(t)$ is:

$$
\begin{align*}
    dX(t) &= (Y(t)b(t, \alpha(t)) - c(t))dt + Y(t)\sigma(t, \alpha(t))dW(t) \\
    &\quad + Y(t)\int_{\mathbb{R}_0} \eta(t, \alpha(t), z)\tilde{N}(dt, dz) \\
    &\quad + Y(t)\gamma(t, \alpha(t))d\tilde{\Phi}(t), \quad t \in [0, T]; \\
    X(t) &= x_0(t), \quad t \in [-\delta, 0] \text{ and } x_0 \in C([-\delta, 0]; \mathbb{R}).
\end{align*}
$$

(4)
Delay – Optimal Consumption from a cash flow

Let $U(t,c,e_i)$, $i = 1, 2, \ldots, D$, be a given stochastic utility function such that

- $t \mapsto U(t,c,e_i)$ is $\mathcal{F}_t$-adapted for each $c \geq 0$ and $e_i \in S$,
- $c \mapsto U(t,c,e_i)$ is $C^1$, $\frac{\partial U}{\partial c}(t,c,e_i) > 0$ for each $e_i \in S$,
- $c \mapsto \frac{\partial U}{\partial c}(t,c,e_i)$ is strictly decreasing for each $e_i \in S$,
- $\lim_{c \to \infty} \frac{\partial U}{\partial c}(t,c,e_i) = 0$ for all $t \in [0,T]$ and $e_i \in S$.

Let $v_0(t,\alpha(t)) := \frac{\partial U}{\partial c}(t,0,\alpha(t))$ and define

$$I(t,v,\alpha(t)) := \begin{cases} 0, & \text{if } v \geq v_0(t,\alpha(t)) \\ \left(\frac{\partial U}{\partial c}(t,\cdot,\alpha(t))\right)^{-1}(v), & \text{if } 0 \leq v < v_0(t,\alpha(t)). \end{cases}$$
Our problem is to find the consumption rate \( \hat{c}(t) \) such that

\[
J(\hat{c}) = \sup_{c \in \mathcal{A}} E \left[ \int_0^T U(t, c(t), \alpha(t)) dt + g(X(T), \alpha(T)) \right]
\]

In this case, the Hamiltonian gets the form:

\[
H(t, x, y, a, e_i, c, p, q, r(\cdot), \omega) = U(t, c, e_i) + (b(t, e_i)y - c)p + y\sigma(t, e_i)q
\]

\[
+ y \int_{\mathbb{R}_0} \eta(t, e_i, z)r(t, z)v(\,dz\,) + y \sum_{j=1}^D \gamma^j(t, e_i)\omega^j(t)\lambda_{ij}
\]
Hence, the corresponding adjoint equations are defined by:

\[
\begin{align*}
dp(t) &= -E[(b(t + \delta, \alpha(t + \delta))p(t + \delta) + \sigma(t + \delta, \alpha(t + \delta))q(t + \delta) \\
&\quad + \int_{\mathbb{R}_0} \eta(t + \delta, \alpha(t + \delta), z)r(t + \delta, z)v(dz) \\
&\quad + \sum_{j=1}^{D} \gamma^j(t, \alpha(t + \delta))\omega^j(t)\lambda_j(t)\chi_{[0,T-\delta]}(t)|\mathcal{F}_t|dt \\
&\quad + q(t)dW(t) + \int_{\mathbb{R}_0} r(t, z)\tilde{N}(dt, dz) + \omega(t)\tilde{\Phi}(t), \ t \in [0, T];
\end{align*}
\]

\[
p(T) = g_x(X(T), \alpha(T)).
\]

Here we observe that maximizing \( H \) w.r.t. \( c \) gives

\[
\frac{\partial U}{\partial c}(t, \hat{c}(t), \alpha(t)) = p(t).
\]
Proposition: Let \( p(t), q(t), r(t, z), \omega(t) \) be the solution of the corresponding ABSDE and suppose that \( 0 \leq p(t) \leq v_0(t, \alpha(t)) \) holds for all \( t \in [0, T] \).
Then the optimal consumption rate \( \hat{c}(t) \) and the corresponding optimal terminal wealth \( X(t) \) are given implicitly by the coupled equations

\[
\hat{c}(t) = I(t, p(t), \alpha(t))
\]

and the equation (4), respectively.
Let us assume that $b(t, \alpha(t)) = b(t)$ is deterministic and $g(X(t), \alpha(t)) = kX(t), \ k > 0$.

By Martingale Representation Theorem for regime-switching jump-diffusions, Crépéy and Matoussi [2008], we can choose $q = r = \omega = 0$.

Then the ABSDE (2)-(3) becomes:

$$
\begin{align*}
    dp(t) &= -b(t + \delta)p(t + \delta)\chi_{[0,T-\delta]}(t)dt, \quad t < T, \\
    p(t) &= k, \quad t \in [T - \delta, T].
\end{align*}
$$
To solve this, we introduce

\[ h(t) = p(T - t), \quad t \in [0, T] \]

Then, we obtain the DDEs:

\[
\begin{align*}
    & dh(t) = b(T - t + \delta)h(t - \delta)dt, \quad t \in [\delta, T], \\
    & h(t) = k, \quad t \in [0, \delta].
\end{align*}
\]

Hence, we can determine \( h(t) \) inductively on each interval as follows:

\[
    h(t) = h(j\delta) + \int_{j\delta}^{t} h'(s)ds = h(j\delta) + \int_{j\delta}^{t} b(T - t + \delta)h(s - \delta)ds \quad (6)
\]

for \( t \in [j\delta, (j + 1)\delta] \).
Proposition: Assume that for each $e_i$, $i = 1, 2, ..., D$, and $t \in [0, T]$, $b(t, e_i) > 0$ is deterministic, $U(t, c, e_i) = \phi(t, e_i) \ln(1 + c)$, where $\phi(t, e_i) > 0$ and $g(x, e_i) = kx$, $k > 0$. Let for each $e_i$, $i = 1, 2, ..., D$, and $t \in [0, T]$, be $h_\delta(T - t) < \phi(t, e_i)$. Then,

$$\hat{c}(t) = \frac{\phi(t, e_i)}{h_\delta(T - t)} - 1,$$

where $h_\delta(\cdot) = h(\cdot)$ is determined by equations (5)-(6).
References

References

Thank you very much for your attention!

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Appendix: Delay – Partial Information

Let $\mathcal{E}_t \subseteq \mathcal{F}_t$, $t \in [0, T]$, be a given subfiltration of $\{\mathcal{F}_t\}_{t \in [0, T]}$ that represents the information available to the controller who decides the value of $u(t)$ at time $t$. For example, $\mathcal{E}_t = \mathcal{F}_{(t-c)+}$ for some given $c > 0$.

Let $U \subset \mathbb{R}$ be a given set of admissible control values $u(t)$, $t \in [0, T]$, and $\mathcal{A}_\mathcal{E}$ be a given family of admissible control processes included in the set of càdlàg, $\mathcal{E}$-adapted, $U$-valued processes such that the defined SHDDEJ, equation (1), has unique solution.
Appendix:
Delay – Partial Information

- **A1** For all \( u \in \mathcal{A}_\varepsilon \) and all bounded \( \beta \) there exists \( \varepsilon > 0 \) such that
  \[ u + s\beta \in \mathcal{A}_\varepsilon \]
  for all \( s \in (-\varepsilon, \varepsilon) \).

- **A2** For all \( t_0 \in [0, T] \) and for all bounded \( \mathcal{E}_{t_0} \)-measurable random variables \( v \), the control process \( \beta(t) \), defined by
  \[ \beta(t) = v\chi_{[t_0, T]}(t), \quad t \in [0, T], \]
  belongs to \( \mathcal{A}_\varepsilon \).

- **A3** For all bounded \( \beta \in \mathcal{A}_\varepsilon \) the derivative process
  \[ \xi(t) := \frac{d}{ds}X^{u+s\beta}(t) \big|_{s=0} \]
  exists.
**Theorem:** Let \( \hat{u} \in \mathcal{A}_\mathcal{E} \) be an optimal control with the corresponding trajectory \((\hat{X}(t), \hat{Y}(t), \hat{A}(t))\) and \((\hat{p}(t), \hat{q}(t), \hat{r}(t, z), \hat{\omega}(t))\) be the unique solution of the corresponding adjoint equation. Under some technical assumptions, the following conditions are equivalent.

(i) \( \frac{d}{ds} J(\hat{u} + s\beta)|_{s=0} = 0 \) for all \( \beta \in \mathcal{A}_\mathcal{E} \).

(ii) \( E \left[ \frac{\partial H}{\partial u} (t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}, \alpha(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{\omega}(t))|_{\mathcal{E}_t} \right]_{u=\hat{u}(t)} = 0 \) a.s. for all \( t \in [0, T] \).
Appendix: Delay – \textbf{Sufficient Maximum Principle under Partial Information}

Theorem: Let \( \hat{u} \in A_\mathcal{E} \) with corresponding state processes \((\hat{X}(t), \hat{Y}(t), \hat{A}(t))\) and suppose there exists an adapted solution \((\hat{p}(t), \hat{q}(t), \hat{r}(t, z), \hat{\omega}(t))\) of the corresponding adjoint equation such that for all \( u \in A_\mathcal{E} \) we have

\[
E \left[ \int_0^T \hat{p}(t)^2 \left( (\sigma(t) - \hat{\sigma}(t))^2 + \int_{\mathbb{R}_0} (\eta(t, z) - \hat{\eta}(t, z))^2 \nu(dz) \right)
+ \sum_{j=1}^D (\gamma_j(t) - \hat{\gamma}_j(t))^2 \lambda_j(t) \right] < \infty
\]

and

\[
E \left[ \int_0^T (X(t) - \hat{X}(t))^2 \left\{ \hat{q}^2(t) + \int_{\mathbb{R}_0} \hat{r}^2(t, z) \nu(dz) + \sum_{j=1}^D (\hat{\omega}_j)^2(t) \lambda_j(t) \right\} dt \right] < \infty.
\]
Furthermore, we assume that the following conditions hold:

1. For almost all $t \in [0, T]$,

\[
E[H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \alpha(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{\omega}(t))|\mathcal{E}_t] = \max_{u \in U} E[H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \alpha(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{\omega}(t))|\mathcal{E}_t].
\]

2. $H$ is a concave function of $x, y, a, u$ for all $(t, e_i) \in [0, T] \times S$.

3. $g(x, e_i)$ is a concave function of $x$ for each $e_i \in S$.

Then $\hat{u}(t)$ is an optimal control process and $\hat{X}(t), \hat{Y}(t), \hat{A}(t)$ are the corresponding controlled state processes.
Appendix:

Bubbles, Jumps and Insiders

Notation:

\((\Omega, \mathcal{F}, P),\) \((\Omega, P) = (\Omega_B \times \Omega_\eta, P_B \times P_\eta),\)

\[
\eta(t) = \int_0^t \int_\mathbb{R} z \tilde{N}(dt, dz) \geq 0, \quad \tilde{N}(dt, dz) = (N - v_\mathcal{F})(dt, dz) = N(dt, dz) - v_\mathcal{G}(dz)dt,
\]

\[
\mathcal{F}_t^B := \sigma\{B(s) \mid s < t, t \in [0, T]\} \vee \mathcal{N}, \quad \mathcal{F}_t^{\tilde{N}} := \sigma\{\tilde{N}(\Delta) \mid \Delta \in \mathcal{B}(\mathbb{R}_0 \times (0, t)), t \in [0, T]\} \vee \mathcal{N},
\]

\[
\mathcal{F}_t = \mathcal{F}_t^B \otimes \mathcal{F}_t^{\tilde{N}},
\]

\[
\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F} \quad \forall t \in [0, T],
\]

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\[ (\Omega, \mathcal{F}, P), \quad (\Omega, P) = (\Omega_B \times \Omega_\eta, P_B \times P_\eta), \]

\[ \eta(t) = \int_0^t \int z \tilde{N}(dt, dz) \geq 0, \quad \tilde{N}(dt, dz) = (N - \nu_\mathcal{F})(dt, dz) = N(dt, dz) - \nu_\mathcal{F}(dz)dt, \]

pure jump Lévy process \quad compensated Poisson random measure \quad Lévy measure

\[ \mathcal{F}^B_t := \sigma\{B(s) \mid s < t, t \in [0, T]\} \vee \mathcal{N}, \quad \mathcal{F}^{\tilde{N}}_t := \sigma\{\tilde{N}(\Delta) \mid \Delta \in \mathcal{B}(\mathbb{R}_0 \times (0, t)), t \in [0, T]\} \vee \mathcal{N}, \]

\[ \mathcal{F}_t = \mathcal{F}^B_t \otimes \mathcal{F}^{\tilde{N}}_t, \]

\[ \mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{I} \quad \forall t \in [0, T], \quad \text{Bubble} \]
Appendix:
Bubbles, Jumps and Insiders

\[
\int_0^\infty \varphi(t, \omega) d^-B(t) = \lim_{\varepsilon \to 0} \int_0^\infty \varphi(t, \omega) \frac{B(t + \varepsilon) - B(t)}{\varepsilon} dt
\]

\[
\int_0^\infty \int \varphi(t, \omega) \tilde{N}(d^-t, dz) = \lim_{n \to \infty} \int_0^\infty \int \varphi(t, z) \tilde{N}(dt, dz)
\]

\[
(U_n \subset \mathbb{R}_0 \text{ compact, } U_n \subseteq U_{n+1} \ (n \in \mathbb{N}), \ \nu_F(U_n) < \infty, \ \bigcup_{n=1}^\infty U_n = \mathbb{R}_0)
\]

\[
X(t) = X(0) + \int_0^t \alpha(s) ds + \int_0^t \beta(s) d^-B(s) + \int_0^\infty \int_0^{\mathbb{R}_0} \gamma(s, z) \tilde{N}(d^-s, dz) \quad \forall t \in [0, T]
\]

\[
d^-X(t) = \alpha(t) dt + \beta(t) d^-B(t) + \int_\mathbb{R}_0 \gamma(t, z) \tilde{N}(d^-t, dz)
\]

Forward Integrals

Forward Process
Appendix:
Bubbles, Jumps and Insiders

If \( Y(t) = f(X(t)) \):

\[
d^{-}Y(t) = \left[ f''(X(t))\alpha(t) + \frac{1}{2} f'''(X(t))\alpha(t) \beta(t)^2 + \int_{\mathbb{R}_0} \{ f(X(t^-) + \gamma(t, z)) \} \right] dt + f'(X(t)) \beta(t) d^{-}B(t) \\
- f(X(t^-)) - f'(X(t^-))\gamma(t, z) \nu_{\mathcal{F}}(dz) \\
+ \int_{\mathbb{R}_0} \left( f(X(t^-) + \gamma(t, z)) - f'(X(t^-)) \right) \tilde{N}(d^{-}t, dz).
\]

Itô Formula for Forward Integrals
Appendix:
Bubbles, Jumps and Insiders

\[ dS_0(t) = r(t)S_0(t)dt, \]
\[ S_0(0) = 1, \]

\[ dS_1(t) = S_1(t^-) \left[ \mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t, z)\tilde{N}(dt, dz) \right], \]
\[ S_1(0) > 0, \]

\[ d^-X^{(c,\pi)}(t) = X^{(c,\pi)}(t^-) \left[ \left\{ r(t) + (\mu(t) - r(t))\pi(t) \right\} dt + \sigma(t)\pi(t)d^-B(t) \right. \]
\[ + \pi(t) \int_{\mathbb{R}_0} \gamma(t, z)\tilde{N}(d^-t, dz) \left. \right] - c(t)dt, \]
\[ X^{(c,\pi)}(0) = v. \]
Appendix:
Bubbles, Jumps and Insiders

If \( \lambda(t) = \frac{c(t)}{X^{(c,\pi)}(t)} \), then

\[
d^- X^{(c,\pi)}(t) = X^{(c,\pi)}(t^-) \left[ \{r(t) - \lambda(t) + (\mu(t) - r(t))\pi(t)\} dt + \sigma(t)\pi(t)d^- B(t) + \pi(t) \int_{\mathbb{R}_0} \gamma(t, z) \tilde{N}(d^- t, dz) \right],
\]

\[
J(\lambda^*, \pi^*) := \sup_{(\lambda, \pi) \in \mathcal{A}} \text{IE} \left[ \int_0^T e^{-\delta(t)} \ln \left( \lambda(t) X^{(c,\pi)}(t) \right) dt + Ke^{-\delta(T)} \ln X^{(c,\pi)}(T) \right].
\]
Assumptions: (1) Uninformed agent has access to filtration $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{G}'_t$ ($t \in [0, T]$),
(2) Lévy measure $\nu_{\mathcal{F}}$ is given by $\nu_{\mathcal{F}}(ds, dz) = \rho \delta_1(dz)ds$ ($\delta_1(dz)$: unit point mass at 1),
(3) $\eta(t)$ is given by $\eta(t) = Q(t) - \rho t$ ($Q$ : Poisson process of intensity $\rho$).

$$
\pi^*_i(t) = \pi^*_h(t) + \frac{\zeta(t)}{\sigma(t)} \\
\pi^*_h(t) = \frac{\mu(t) - r(t)}{\sigma(t)^2} - \frac{\rho}{\sigma(t)^2} \\
\lambda^*_i(t) = \lambda^*_h(t) = \frac{e^{-\delta(t)}}{T} \int_t e^{-\delta(s)} ds + Ke^{-\delta(T)}
$$

$$
\zeta(t) = \frac{1}{2\sigma(t)} \left[ -\mu(t) + r(t) + \rho + \sigma(t)\alpha(t) - \sigma(t)^2 \right. \\
\left. + \sqrt{\left( \mu(t) - r(t) - \rho + \sigma(t)\alpha(t) + \sigma(t)^2 \right) + 4\sigma(t)^2 \theta(t) } \right],
$$

$$
\alpha(t) = \frac{\text{IE}\left[ B(T_0) - B(t) \mid \mathcal{G}_t \right]}{T_0 - t} \\
\theta(t) = \frac{\text{IE}\left[ Q(T_0) - Q(t) \mid \mathcal{G}_t \right]}{T_0 - t}
$$
Mixed Case

\[ J_i(c_{\lambda_i}^*, \pi_i^*) = J_h(c_{\lambda_h}^*, \pi_h^*) + \text{IE} \left[ \int_0^T e^{-\delta(t)} \int_0^t \left( -\frac{1}{2} \xi(s)^2 + \xi(s)\alpha(s) \right) ds \, dt \right] \]

\[ + \text{IE} \left[ \int_0^T e^{-\delta(t)} \int_0^t \int \ln \left( 1 + \frac{z\xi(s)\sigma(s)}{\sigma(s)^2 + (\mu(s) - r(s) - \rho)z} \right) v \varphi(ds, dz) \, dt \right] \]

\[ + K \text{IE} \left[ e^{-\delta(T)} \int_0^T \left( -\frac{1}{2} \xi(s)^2 + \xi(s)\alpha(s) \right) ds \right] \]

\[ + K \text{IE} \left[ e^{-\delta(T)} \int_0^T \int \ln \left( 1 + \frac{z\xi(s)\sigma(s)}{\sigma(s)^2 + (\mu(s) - r(s) - \rho)z} \right) v \varphi(ds, dz) \right] \]

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For any \((s, y, i) \in [0, T) \times \mathbb{R}^N \times S\), consider:

- **State equation**
  \[
  dX(t) = f(t, X(t^-), \alpha(t^-), u(t^-)) \, dt + g(t, X(t^-), \alpha(t^-), u(t^-)) \, dW(t) \\
  + \int_{\mathbb{R}_0^K} h(t, X(t^-), \alpha(t^-), u(t^-), z) \tilde{J}(dt, dz) , \quad t \in [s, T].
  \]

- **Objective functional**
  \[
  J(s, y, i; u(\cdot)) = E \left[ \int_s^T L(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)) \, dt \\
  + \psi(T, X_{s,y,i}(T; u(\cdot)), \alpha_{s,i}(T)) \right].
  \]
For each \( s \in [0, T) \), denote by \( \mathcal{U}^w[s, T] \) the set of weak admissible controls:

- 8-tuples of the form

\[
\tilde{u} = (\Omega, \mathcal{F}, \mathbb{F}, P, \mathcal{W}(\cdot), \alpha(\cdot), \eta(\cdot), u(\cdot)).
\]

Optimal control problem in dynamic programming form:

- For any \((s, y, i) \in [0, T) \times \mathbb{R}^N \times S\), find \(\tilde{u}(\cdot) \in \mathcal{U}^w[s, T]\) such that

\[
J(s, y, i; \tilde{u}(\cdot)) = \sup_{u(\cdot) \in \mathcal{U}^w[s, T]} J(s, y, i; u(\cdot)).
\]

The value function \( V : [0, T] \times \mathbb{R}^N \times S \to \mathbb{R} \) is well-defined by

\[
\begin{cases}
V(s, y, i) = \sup_{u(\cdot) \in \mathcal{U}^w[s, T]} J(s, y, i; u(\cdot)) \\
V(T, y, i) = \psi(T, y, i)
\end{cases}
\]

\((s, y, i) \in [0, T) \times \mathbb{R}^N \times S\).
Appendix:
Dynamic Programming Principle

Theorem (Bellman's Principle of Optimality)

Assume that conditions (A1)-(A2) hold. Then, for any \((s, y, i) \in [0, T) \times \mathbb{R}^N \times S\) we have that

\[
V(s, y, i) = \sup_{u \in U^w[s, T]} E \left[ \int_s^{\hat{s}} L(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)) \, dt + V(\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s})) \right]
\]

for all \(\hat{s} \in [s, T]\).
The dynamic programming principle can be used to derive a HJB equation:
- partial integro-differential equation whose “solution” is the value function of the optimal control problem.

**Theorem (Hamilton-Jacobi-Bellman Equation)**

Suppose that conditions (A1)-(A2) hold and that the value function \( V \) is such that \( V(\cdot, \cdot, \alpha) \in C^{1,2} ([0, T) \times \mathbb{R}^N; \mathbb{R}) \) for every \( \alpha \in S \). Then, for each \( \alpha \in S \), the value function \( V(\cdot, \cdot, \alpha) \) defined on \( [0, T) \times \mathbb{R}^N \) is the solution of the Hamilton-Jacobi-Bellman equation:

\[
\begin{aligned}
V_t + \sup_{u \in U} H(t, X, \alpha, u, V, V_X, V_{XX}) &= 0, \\
V(T, X, \alpha) &= \psi(T, X, \alpha), \quad (t, X, \alpha) \in [0, T) \times \mathbb{R}^N \times S.
\end{aligned}
\]
Theorem (Hamilton-Jacobi-Bellman Equation)

where the Hamiltonian function $\mathcal{H}(t, X, \alpha, u, V, V_X, V_{XX})$ is given by

$$
\mathcal{H}(t, X, \alpha, u, V, V_X, V_{XX}) = L(t, X, \alpha, u) + \langle V_X(t, X, \alpha), f(t, X, \alpha, u) \rangle \\
+ \frac{1}{2} \text{tr} \left( g^T(t, X, \alpha, u) V_{XX}(t, X, \alpha) g(t, X, \alpha, u) \right) \\
+ \sum_{j \in S : j \neq \alpha} q_{\alpha j} (V(t, X, j) - V(t, X, \alpha)) \\
+ \sum_{k=1}^{K} \int_{\mathbb{R}^1_0} W_k(t, X, \alpha, u, V, V_X, z_k) \nu_k(\text{d}z_k)
$$

and

$$
W_k(t, X, \alpha, u, V, V_X, z_k) = V(t, X + h^{(k)}(t, X, \alpha, u, z_k), \alpha) - V(t, X, \alpha) \\
- \langle V_X(t, X, \alpha), h^{(k)}(t, X, \alpha, u, z_k) \rangle.
$$