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Finance, Economics and Nature: Stochastic Optimal Control under Shifts, Switches and Impulses

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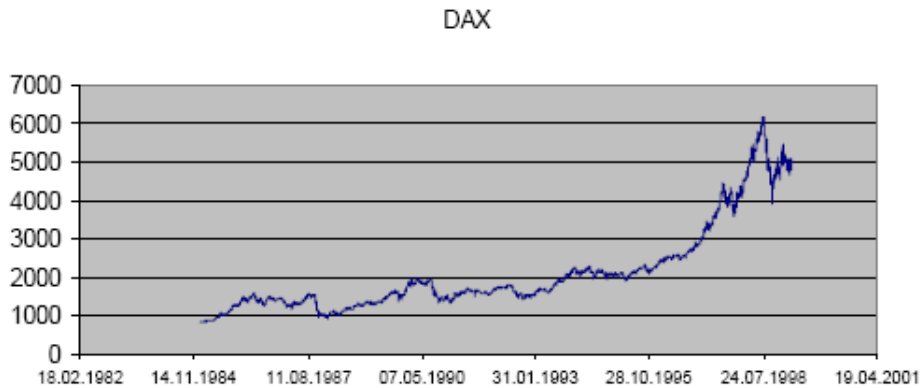
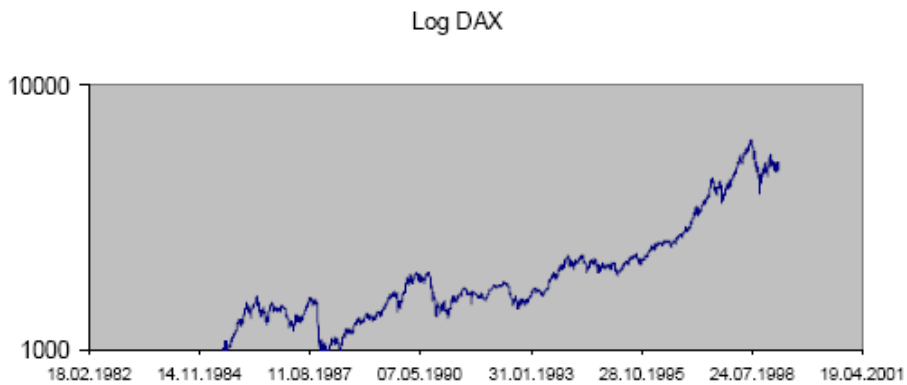
Outline

- Introduction
- Stock Price Dynamics, and Lévy Processes
- Stochastic Hybrid Systems (with Jumps) and Markov Switching Models
- Stochastic Control of Hybrid Systems I and II
- Optimal Control of Stochastice Hybrid Systems with Jumps: 2 Approaches
- Example
- … under Delay: Maximum Principle
- Example
- Appendix: … and under Partial Information
- Appendix: Insider Information; Dynamic Programming and HJB Eqn.

Stock Price Dynamics



$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t$$



d-fine, 2002

Stock Price Dynamics

NIG Lévy Asset Price Model

Normal Inverse Gaussian Process (NIG) is the subclass of generalized hyperbolic laws and has the following representation:

$$f_{NIG}(x;\alpha,\beta,\delta,\mu)=\alpha\exp\left\{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\right\}\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}},$$

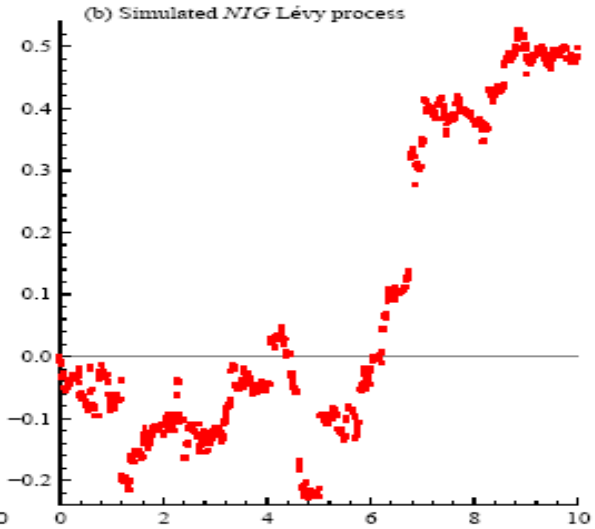
where

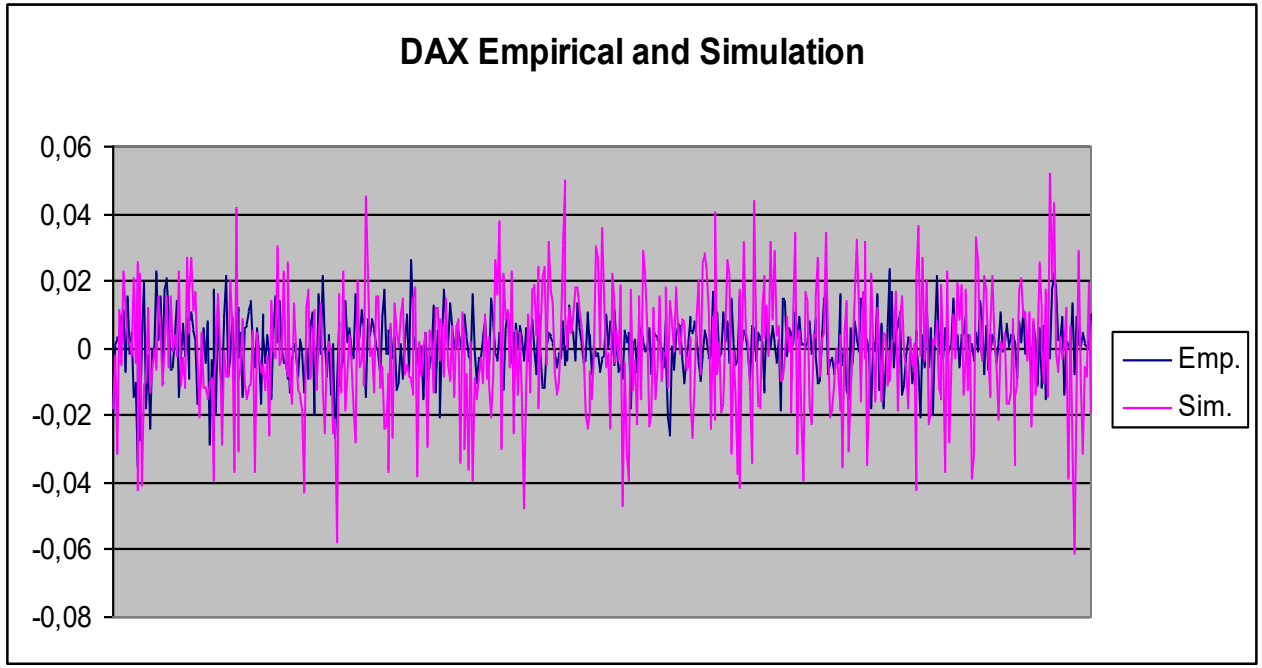
$$K_1(y)=\frac{y}{4}\int_0^\infty \exp(t+\frac{y^2}{4t})t^{-2}dt.$$

Ö. Önalan, 2006

Stock Price Dynamics

NIG Lévy Asset Price Model





Lévy Processes

Definition: A cadlag adapted processes

$$L = (L_t \mid t \geq 0)$$

defined on a probability space (Ω, *F*, *P*) is said to be a ***Lévy processes*****,** if it possesses the following properties:

Lévy Processes

(í) $P\left(L_0 = 0 \right) = 1.$

(íí) For $0 \le s \le t,\ L_t - L_s$ is independent of F_s,
i.e., L has independent increments.

(ííí) For any $0 \le s \le t,\ L_t - L_s$ is equal in distribution to L_{t-s}
(the distribution of $L_{t+s} - L_s$ does not depend on t);
L has stationary increments.

(ív) For every $s, t \ge 0$ and $\varepsilon > 0$,
$$\lim_{s \to t} P\left(\left| L_t - L_s \right| > \varepsilon \right) = 0,$$
i.e., L is stochastically continuous.

Lévy Processes

- There is strong interplay between

  Lévy processes and infinitely divisible distributions.

Proposition: If L is a Lévy processes, then L_t is infinitely divisible for each $t > 0$.

Proof : For any $n \in \mathbb{N}$ and any $t \geq 0$:

$$L_t = L_{t/n} + \left(L_{2t/n} - L_{t/n} \right) + \ldots + \left(L_t - L_{(n-1)t/n} \right).$$

- Together with the stationarity and independence of increments we conclude that the random variable L_t is infinitely divisible.

Lévy Processes

- Moreover, for all $u \in \mathbb{R}$ and all $t \geq 0$ we define

$$\Psi_t(u) := \ln E\left[\, e^{i\,u\,L_t} \,\right];$$

hence, for rational $t > 0$:

$$t\,\Psi_1(u) = \Psi_t(u).$$

Lévy Processes

For every Lévy process, the following property holds:

$$E[e^{iuL_t}] = e^{t\Psi_1(u)}$$

$$= \exp[t(ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{iux} - 1 - iux1_{\{|x|<1\}})\nu(dx))]$$

where $\Psi(u) = \Psi_1(u)$ is the characteristic exponent of $L_1 = X$.

Lévy Processes

- The triplet (b,c,ν) is called the Lévy or characteristic triplet and

$$\Psi(u) = ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{iux} - 1 - iux1_{\{|x|<1\}})\nu(dx)$$

is called the Lévy or characteristic exponent,

- where $b \in \mathbb{R}$: drift term,

  $c \in \mathbb{R}^+$: diffusion coefficient and

  ν : Lévy measure.

Lévy Processes

- Another key concept, the *Lévy-Ito Decomposition Theorem*, allows one to describe the structure of a Lévy process sample path.

  Lévy-Ito Decomposition Theorem:

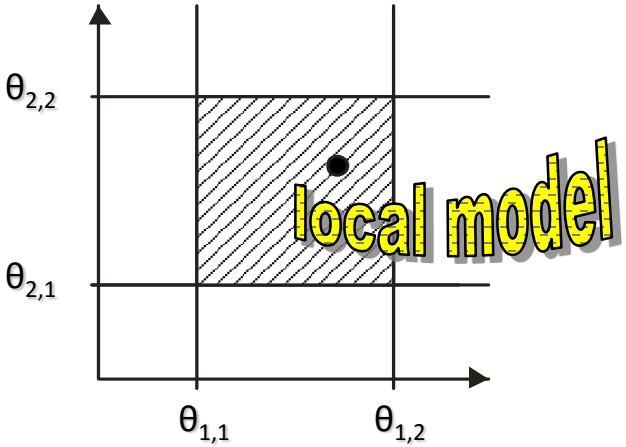
  Let X be a Lévy process, the distribution of X_1 parametrized by (β, σ^2, ν).

  Then X decomposes as $X_t = \beta t + \sigma W_t + J_t + M_t$, where W_t is a Brownian motion, and $\Delta X_t = X_t - X_{t-}$ $(t \geq 0)$ is an independent Poisson point process with intensity measure ν, $J_t = \sum_{s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq 1\}}$ and M_t is a martingale with jumps, $\Delta M_t = \Delta X_t 1_{\{|\Delta X_t| < 1\}}$.

Hybrid Systems

Basic Idea

$$X(k+1) = \mathrm{M}_{s(k)} X(k) + C_{s(k)}$$

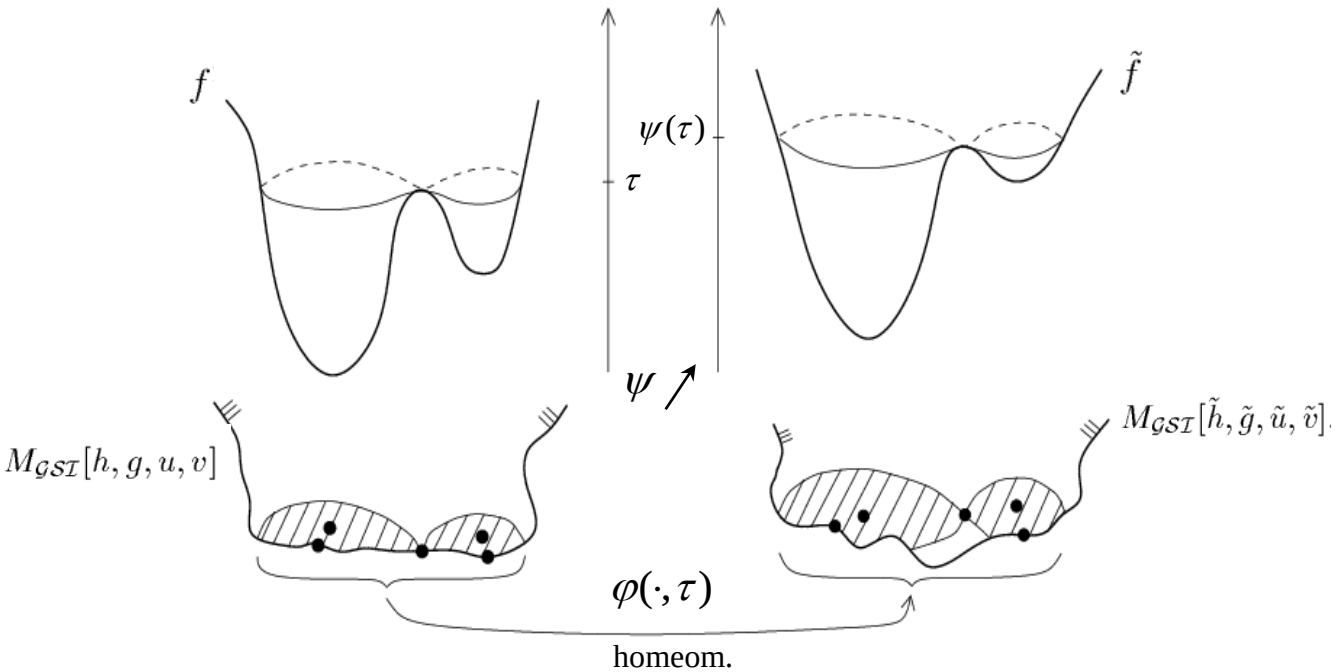


$$s(k) := F_B Q(X(k-1))$$

$$Q_i(X(k)) := \begin{cases} 1 & \text{if } X_i(k) > \theta_i \\ 0 & \text{otherwise} \end{cases}$$

Hybrid Systems

$$\exists\ \mathcal{O}_{(f,h,g,u,v)}\ \forall\ (\tilde{f},\tilde{h},\tilde{g},\tilde{u},\tilde{v}) \in \mathcal{O}_{(f,h,g,u,v)} \qquad \exists\ \ \psi(\cdot),\varphi(\cdot,\cdot)\in C^0:$$



Hybrid Systems

$$\dot{E} = M(E)E$$

M : ***constant*** (*nxn*)-matrix with entries m_{ij} representing the effect which the expression level of gene j has on the change of expression of gene i

genetic regulation network

mixed-integer nonlinear optimization problem (MINLP):

$$\min_{M=(m_{ij})} \sum_{k=1}^{l} \left\| M\bar{E}^{(k)} - \dot{\bar{E}}^{(k)} \right\|_2^2,$$

$i, j \in G$
$G = \{1, ..., n\}$

subject to

$$\begin{cases} m_{ij} \geq \begin{cases} -\lambda(i), & i = j, \\ 0, & i \neq j. \end{cases} \\ (1 - y_{ij}) \cdot m_{ij} = 0, \quad \forall\, i, j \in G. \\ \sum_{j \in G} y_{ij} \leq deg_{max,i} \quad , \quad \forall\, i \in G. \end{cases}$$

Binary variables $y_{ij} \in \{0, 1\}$:

$$y_{ij} = \begin{cases} 0, & \text{if} \quad m_{ij} = 0, \\ 1, & \text{if} \quad m_{ij} \neq 0. \end{cases}$$

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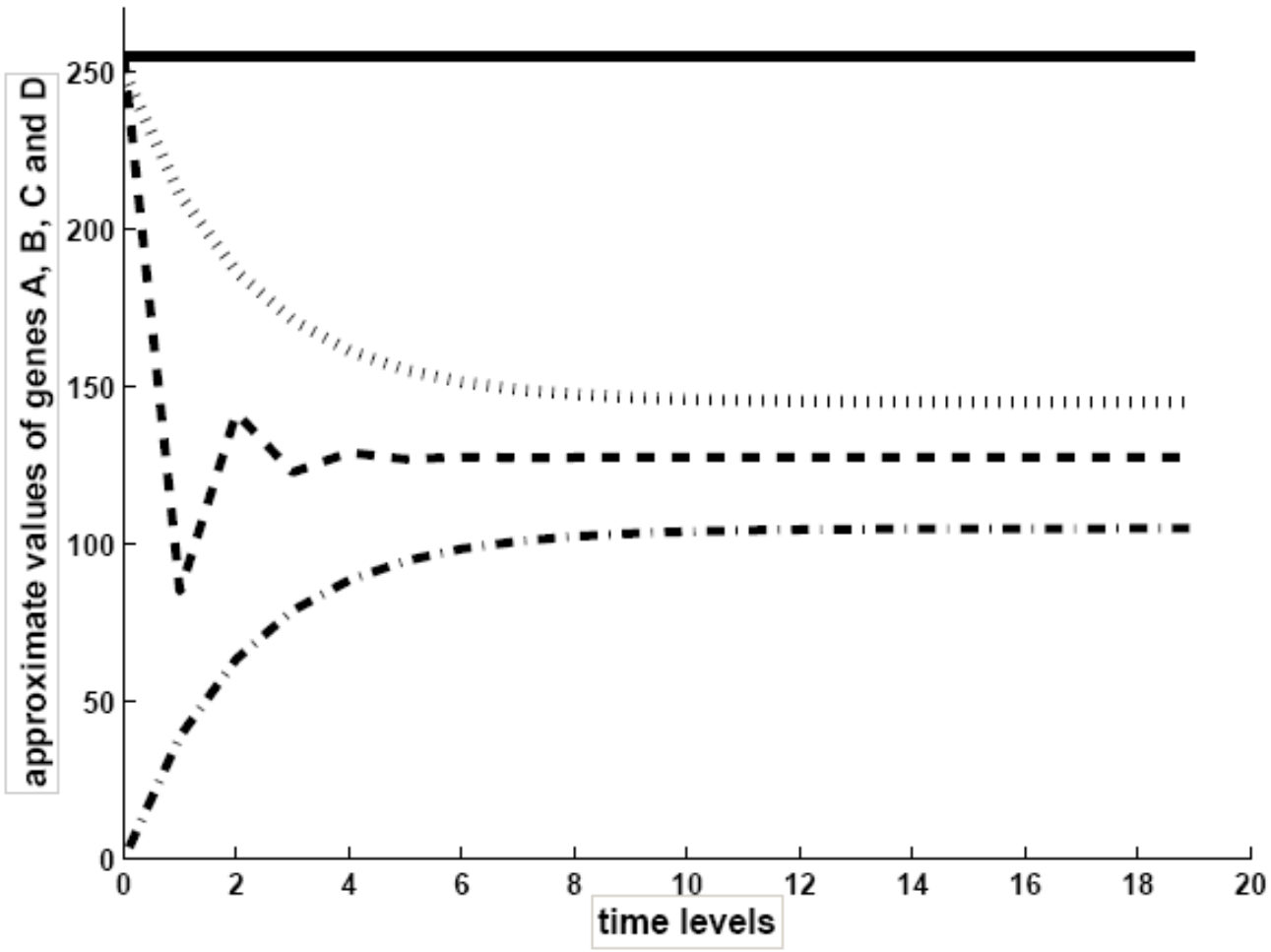
Results of 3rd-order Heun Method for all genes:

____ gene A
..... gene B
_ . _ . gene C
- - - - gene D



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Stochastic Hybrid Model I

$$\left(X(t),\theta(t)\right)_{t\in[0,\infty)}:$$

$$X(t)\in\mathbb{R}^d,\ \theta(t)\in\Theta=\{0,1,\ldots,N\},\quad \text{where}$$

- $\theta(t)$ is a pure jump process taking values in Θ,
- $\left(X(t),\theta(t)\right)$ is a switching jump-diffusion between jumps.

M.K. Ghosh and A. Bagchi, 2004

Stochastic Control of Hybrid Systems

The ***asset price*** dynamics is given by the following system:

$$(1)\quad \begin{cases} dX(t) = \mu(X(t),\theta(t))dt + \sigma(X(t),\theta(t))dW(t) + \int_{\mathbb{R}} g(X(t^-),\theta(t^-),u)N(dt,du), \\ d\theta(t) = \int_{\mathbb{R}} h(X(t^-),\theta(t^-),u)N(dt,du), \end{cases}$$

$$\text{for} \quad t \geq 0,$$

$$X(0) = X_0, \;\; \theta(0) = \theta_0 .$$

The price process $X(t)$ switches from one jump-diffusion path to another as the discrete component $\theta(t)$ moves from one step to another.

Stochastic Control of Hybrid Systems

$h:\mathbb{R}^d \times \Theta \times \mathbb{R} \to \mathbb{R}^N$ is defined as

$$h(x,i,u) = \begin{cases} j-i, & \text{if} \quad u \in \Delta_{ij}(x), \\ 0, & \text{otherwise,} \end{cases}$$

$N(\cdot)$: Poisson random measure,

$\tilde{N}(\cdot)$: Poisson process on $(\Omega, \mathscr{F}, P)$ corresponding to the given Poisson random measure,

$D_{\tilde{N}}$: domain of $\tilde{N}(\cdot)$,

$D = \left\{ t \in D_{\tilde{N}} \mid \tilde{N}(t) \in U \right\}$,

$\mathscr{F}_t = \sigma\{W(s), N(A,B) \mid \ s \le t, \ A \in \mathscr{B}([0,T]), B \in \mathscr{B}(\mathbb{R})\}$

Merton's Consumption Investment Problem

An investor with a finite lifetime must determine the amounts c_t that will be consumed and the fraction of wealth π_t that will be invested in a stock portfolio,

so as to **maximize *expected lifetime utility***.

Assuming a *relative consumption rate*

$$\lambda(t) = \frac{c(t)}{X^{(c,\pi)}(t)},$$

the ***wealth process*** evolves with the following SDE:

$$d^{-}X^{(\lambda,\pi)}(t,\theta_t) = X^{(\lambda,\pi)}(t,\theta_t)\Big[\Big\{r(t,\theta_t) - \lambda(t,\theta_t) + (\mu(X_t^{(\lambda,\pi)},\theta_t) - r(t,\theta_t))\pi(t,\theta_t)\Big\}dt$$

$$+\sigma(X_t^{(\lambda,\pi)},\theta_t)\pi(t,\theta_t)d^{-}B(t) + \pi(t,\theta_t)\int_{\mathbb{R}_0} g(X_{t^-}^{(\lambda,\pi)},\, \theta_{t^-},\, u)N(dt^-,du)\Big].$$

D. David and Y. Yolcu Okur, 2009

Merton's Consumption Investment Problem

We assume a ***constant relative risk aversion*** **(CRRA) utility function**

$$U(x,t) = \frac{x^{1-\gamma}}{1-\gamma},$$

where $\gamma \in (0,\infty) \setminus \{1\}$.

Hence, the objective takes the following form

$$\max_{(c,\pi)} E\left[\int_0^T e^{-\delta s} U(c(s),s)\,ds + e^{-\delta T} U(X^{(c,\pi)}(T),T)\right] \qquad (2)$$

$$\text{subject to } c(t) \geq 0,\ X^{(c,\pi)}(t) > 0\ (t \in [0,T]),\ X^{(c,\pi)}(0) = \upsilon.$$

Hamilton-Jacobi-Bellman Equation

In applying the ***dynamic programming*** approach, we solve the Hamilton-Jacobi-Bellman (HJB) equation associated with the utility maximization problem (2).

From (W. Fleming and R. Rishel, 1975) we have that the corresponding **HJB equation** is given by

$$\rho J(\upsilon,t) = \max_{c_t \geq 0,\, \pi_t \in \mathbb{R}^n} \Big\{ U(c_t) + J_t(\upsilon,t) + J_\upsilon(\upsilon,t)\Big(\upsilon\big[\pi_t^T(\mu - r) + r\big] - c_t\Big) + \frac{1}{2} J_{\upsilon\upsilon}(\upsilon,t)\upsilon^2 \pi_t^T \sigma\sigma^T \pi_t \Big\} \qquad (3)$$

subject to the terminal condition $J(\upsilon,T) = u(\upsilon)$,
where J the value function is given by

$$J(\upsilon,t) = \sup_{(c,\pi)} E\left[\int_t^T e^{-\delta s} U(c_s,s)\,ds + e^{-\delta T} U(X^{c,\pi}(T),T)\right].$$

Hamilton-Jacobi-Bellman Equation

In solving the HJB equation (3), the static optimization problem

$$\max_{c_t \geq 0,\, \pi_t \in \mathbb{R}^n} \left\{ U(c_t) + J_\upsilon(\upsilon,t)\left(\upsilon\left[\pi_t^T(\mu - r) + r\right] - c_t\right) + \frac{1}{2} J_{\upsilon\upsilon}(\upsilon,t)\upsilon^2 \pi_t^T \sigma\sigma^T \pi_t \right\}$$

can be handled separately to reduce the HJB equation (3) to a nonlinear partial differential equation of J.

Introducing the Lagrange function as

$$L\left(\pi(\upsilon,t), c(\upsilon,t), \tilde{\lambda}(\upsilon,t)\right) = J_\upsilon(\upsilon,t)\left(\upsilon\left[\pi_t^T(\mu - r) + r - c_t\right]\right)$$
$$+ \frac{1}{2}\upsilon^2 \left\|\pi_t^T \sigma\right\|_2^2 J_{\upsilon\upsilon}(\upsilon,t) + u(c_t) - \tilde{\lambda}(\upsilon,t) c_t, \qquad (4)$$

where $\tilde{\lambda}$ is the Lagrange multiplier.

Hamilton-Jacobi-Bellman Equation

Simultaneous resolution of these first-order conditions yields the optimal solutions π^{opt}, c^{opt} and $\tilde{\lambda}^{opt}$.

Substituting these into (3) gives the PDE

$$-\delta J(\upsilon,t)+\frac{\left(c^{opt}(\upsilon,t)\right)^{1-\gamma}}{1-\gamma}+J_t(\upsilon,t)+J_\upsilon(\upsilon,t)\left(\upsilon\left[\left(\pi^{opt}(\upsilon,t)\right)^T(\mu-r)+r\right]-c^{opt}(\upsilon,t)\right)$$
$$+\frac{1}{2}J_{\upsilon\upsilon}(\upsilon,t)\upsilon^2\left(\pi^{opt}(\upsilon,t)\right)^T\sigma\sigma^T\left(\pi^{opt}(\upsilon,t)\right)=0, \qquad (5)$$

with terminal condition

$$J(\upsilon,T)=e^{-\delta T}\frac{\upsilon^{1-\gamma}}{1-\gamma},$$

which can then be solved for the optimal value function $J^{opt}(\upsilon,t)$.

Hamilton-Jacobi-Bellman Equation

Because of the nonlinearity in π^{opt} and c^{opt},
the first-order conditions together with the HJB equation are a nonlinear system.

So (3) and (5) do not always have an analytic solution and numerical methods such as *Newton's method* or *Sequential Quadratic Programming* (SQP) are required to solve for
$\pi^{opt}(\upsilon,t)$, $c^{opt}(\upsilon,t)$, $\tilde{\lambda}^{opt}(\upsilon,t)$ and $J^{opt}(\upsilon,t)$
iteratively.

D. Akume and G.-W. Weber, 2010

Stochastic Hybrid Model II

$$dX(t) = b^n(X(t), \theta(t))dt + \sigma^n(X(t), \theta(t))dW^n(t),$$

$$P(\theta(t+\delta t) = j \mid \theta(t) = i,\ X(s),\ \theta(s), s \le t) = \lambda_{ij}^n(X(t))\delta t + \mathrm{O}(\delta t) \quad \forall\, i \neq j,$$

$$X(0) = x_0,\ \ \theta(0) = \theta_0$$

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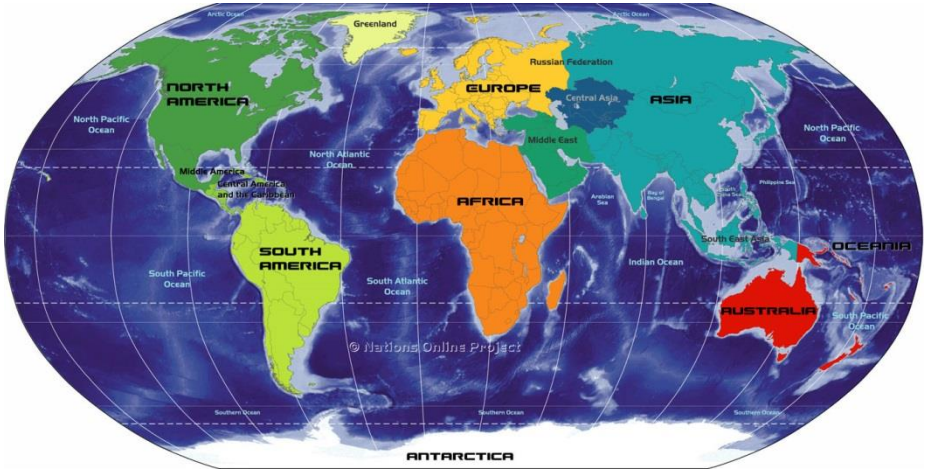

Stochastic Hybrid Model II

$$\mathscr{F}_t^n = \sigma\{W^n(s), N(A,B) \mid \ 0 \le s \le t, \ A \in \mathscr{B}([0,t]), B \in \mathscr{B}(\mathbb{R})\},$$

$$\tau_n = \inf\{t \ge 0 \mid \ X(t) \in A_n\},$$

$$\mathscr{F}_t = \bigvee_{n \in \mathrm{IN}} \mathscr{F}_t^n,$$

$$0 = \tau_0 = \tau_1 < \tau_2 < \ldots < \tau_m < \ldots, \qquad \tau_m \to \infty \ \ (m \to \infty),$$

$$\eta(t) = n, \quad \text{if} \ \ (X(t), \theta(t)) \in S_n \times \Theta_n$$

Stochastic Hybrid Model II

$$dX(t) = \left(b(X(t),\theta(t),\eta(t)) + \sum_{m=0}^{\infty} \left(\tilde{g}_1(X(\tau_m^-),\theta(\tau_m^-),\eta(\tau_m^-)) - X(\tau_m^-) \right) \delta(t-\tau_m) \right) dt$$
$$+ \sigma(X(t),\theta(t),\eta(t)) dW^{\eta(t)}(t),$$

$$d\theta(t) = \int_{\mathbb{R}} h(X(t^-),\theta(t^-),\eta(t^-),u) N(dt,du)$$
$$+ \sum_{m=0}^{\infty} \left(\tilde{g}_2(X(\tau_m^-),\theta(\tau_m^-),\eta(\tau_m^-)) - \theta(\tau_m^-) \right) \delta(t-\tau_m) dt,$$

$$d\eta(t) = \sum_{m=0}^{\infty} \left(\tilde{h}(X(\tau_m^-),\theta(\tau_m^-),\eta(\tau_m^-)) - \eta(\tau_m^-) \right) I_{\{\tau_m \le t\}}$$

Stochastic Control of SHS with Jumps

(a) ***Discrete Approximation:*** B.-Z. Temocin, G.-W. Weber

(b) ***Closed-Form Solutions:*** N. Azevedo, D. Pinheiro, G.-W. Weber

Stochastic Control of SHS with Jumps

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Dynamic programming for a Markov-switching jump-diffusion

JCAM 267: 1–19 (2014)

Merton's Consumption Investment Problem

- Setup:
  - Standard one-dimensional Brownian motion $W = (W(t) : t \in [0, T])$.
  - One-dimensional Lévy process $(\eta(t) : t \in [0, T])$ with Poisson random measure $J(t, A)$ with finite intensity $\nu(A) = E[J(1, A)]$.
  - Continuous time Markov process $(\alpha(t) : t \in [0, T])$ with a finite space state $S = \{a_1, \ldots, a_n\}$ and generator $Q = (q_{ij})_{i,j \in S}$.
  - $W(\cdot), \eta(\cdot), \alpha(\cdot)$ independent.

- Risk-free asset:

$$\mathrm{d}S_0(t) = r(t, \alpha(t_-))S_0(t_-)\mathrm{d}t \ , \qquad S_0(0) = s_0 > 0.$$

- Risky asset:

$$\begin{aligned} \mathrm{d}S_1(t) &= \mu(t, \alpha(t_-))S_1(t_-)\mathrm{d}t + \sigma(t, \alpha(t_-))S_1(t_-)\mathrm{d}W(t) \\ &\quad + S_1(t_-)\int_{\mathbb{R}_0^1} h(t, \alpha(t_-), z)\tilde{J}(\mathrm{d}t, \mathrm{d}z) \ , \qquad S_1(0) = s_1 > 0. \end{aligned}$$

Merton's Consumption Investment Problem

Control variables

- *consumption process* $\{c(t) : t \in [0, T]\}$.
- *portfolio process* $\{\Theta(t) : t \in [0, T]\}$

$$\Theta(t) = (\theta_0(t), \theta_1(t)) \in [0,1] \times [0,1] \ ,$$

where

$$\theta_0(t) + \theta_1(t) = 1 \ , \qquad 0 \leq t \leq T \ .$$

State variable

- *wealth process* $\{X(t) : t \in [0, T]\}$

$$\mathrm{d}X(t) = \Big(-c(t) + \Big(\theta_0(t) r(t, \alpha(t_-)) + \theta_1(t)\mu(t, \alpha(t_-))\Big) X(t_-)\Big)\,\mathrm{d}t$$
$$+ \theta_1(t) X(t_-) \left(\sigma(t, \alpha(t_-))\mathrm{d}W(t) + \int_{\mathbb{R}_0^1} \eta(t, \alpha(t_-), z)\tilde{J}(\mathrm{d}t, \mathrm{d}z)\right) .$$

Power Utility Functions

- We assume that the utility functions belong to the class of constant relative risk aversion (CRRA).
- Let

$$U(t,c,a) = \mathrm{e}^{-\rho t}\frac{c^{\gamma_a}}{\gamma_a}\,, \qquad \Psi(t,x,a) = \mathrm{e}^{-\rho t}\frac{x^{\gamma_a}}{\gamma_a}$$

  where
  - $\gamma_a \in (0,1)$ is the risk aversion coefficient associated with the state of the Markov process $\alpha(t) = a \in S$;
  - $\rho > 0$ is the discount rate.
- Let $F : [0,1] \times [0,T] \times S \to \mathbb{R}$ be given by

$$\begin{aligned} F(\theta;t,a) &= \gamma_a\left[r(t,a) + \theta\big(\mu(t,a) - r(t,a)\big) - \frac{1}{2}(1-\gamma_a)\theta^2\sigma^2(t,a)\right] \\ &\quad + \int_{\mathbb{R}_0^1}\Big((1+\theta\eta(t,a,z))^{\gamma_a} - 1 - \gamma_a\theta\eta(t,a,z)\Big)\nu(\mathrm{d}z). \end{aligned}$$

Power Utility Functions

Theorem (Power utility functions)

The maximum expected utility is given by

$$V(t,x,a) = \xi_a(t)\frac{x^{\gamma_a}}{\gamma_a}$$

and the corresponding optimal strategies are of the form

$$c^*(t,x,a) = x(e^{\rho t}\xi_a(t))^{-1/(1-\gamma_a)}$$

and

$$\theta^*(t,a) = \begin{cases} 1 & \text{if} \quad \mu(t,a) > r(t,a) \quad \text{and} \quad F'(1;t,a) \geq 0 \\ \hat{\theta}(t,a) & \text{if} \quad \mu(t,a) > r(t,a) \quad \text{and} \quad F'(1;t,a) < 0 \\ 0 & \text{if} \quad \mu(t,a) \leq r(t,a) \end{cases}$$

where $\hat{\theta}(t,a)$ is the unique solution of $F'(\theta;t,a) = 0$ in $(0,1)$ and $\xi_a(t)$, $a \in S$, are the solutions of the following coupled ordinary differential equations terminal value problem

$$\begin{aligned} \xi_a'(t) \quad + \quad (1-\gamma_a)e^{-\rho t/(1-\gamma_a)}\xi_a(t)^{-\gamma_a/(1-\gamma_a)} + F(\theta^*(t,a);t,a)\xi_a(t) \\ + \sum_{j\in S: j\neq\alpha} q_{aj}(\xi_j(t) - \xi_a(t)) = 0 \\ \xi_a(T) \quad = \quad e^{-\rho T} \ . \end{aligned}$$

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A simple example

- Consider a finite space state $\mathcal{S} = \{+, 0, -\}$.
- A reasonable choice of parameters should satisfy:
  1) $r_+ > r_0 > r_- \geq 0$;
  2) $\mu_+ > \mu_0 > \mu_-$;
  3) $\mu_+ > r_+, \quad \mu_0 \approx 0, \quad \mu_- < 0$;
  4) $\sigma_+ < \sigma_0 < \sigma_-$.
- one risk-free security and one risky asset driven by a jump diffusion market with regime-switching.

A simple example

We made the following choice of parameters:

- $T = 20$.
  - (i) State $+$: $r_+ = 0.05$, $\mu_+ = 0.20$ and $\sigma_+ = 0.11$.
  - (ii) State 0: $r_0 = 0.03$, $\mu_0 = 0.04$ and $\sigma_0 = 0.21$.
  - (iii) State -: $r_- = 0.01$, $\mu_- = -0.1$ and $\sigma_- = 0.40$.
- In what concerns the jump process, we take $h(t, a, z) = z$. Let
  - $\nu(\mathrm{d}z)$ be the measure associated with a uniform distribution with support $[-0.25, 0.25]$;
  - the waiting time between jumps be exponentially distributed with mean $\lambda = 1$.
- The generator of the Markov process $\alpha(\cdot)$, $Q = (q_{ij})_{i,j \in S}$, is determined by $q_{+0} = 3$, $q_{+-} = 6$, $q_{0+} = 1$, $q_{0-} = 1$, $q_{-+} = 2$ and $q_{-0} = 1$.
- CRRA utilities with risk aversion parameters $\gamma_+ = 0.6$, $\gamma_0 = 0.5$ and $\gamma_- = 0.4$
- $\rho = 0.03$.

A simple example

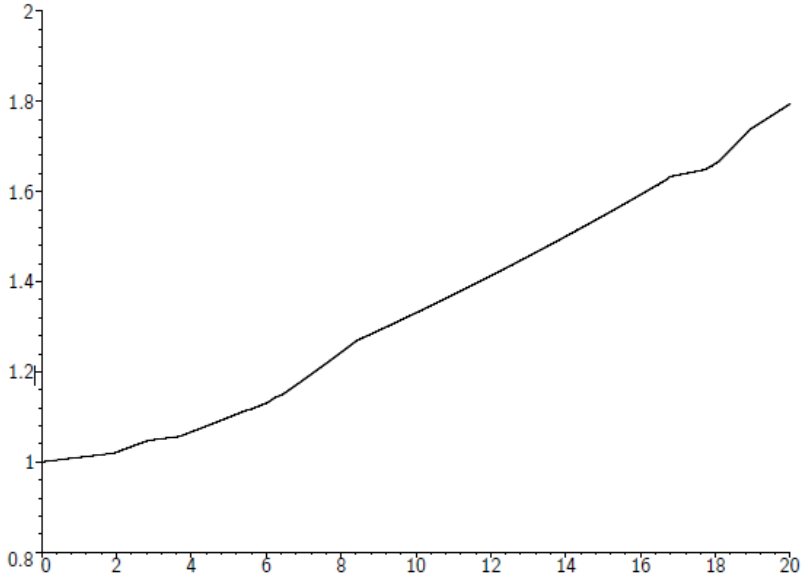

Figure: Sample path of the risk free asset price evolution with time.

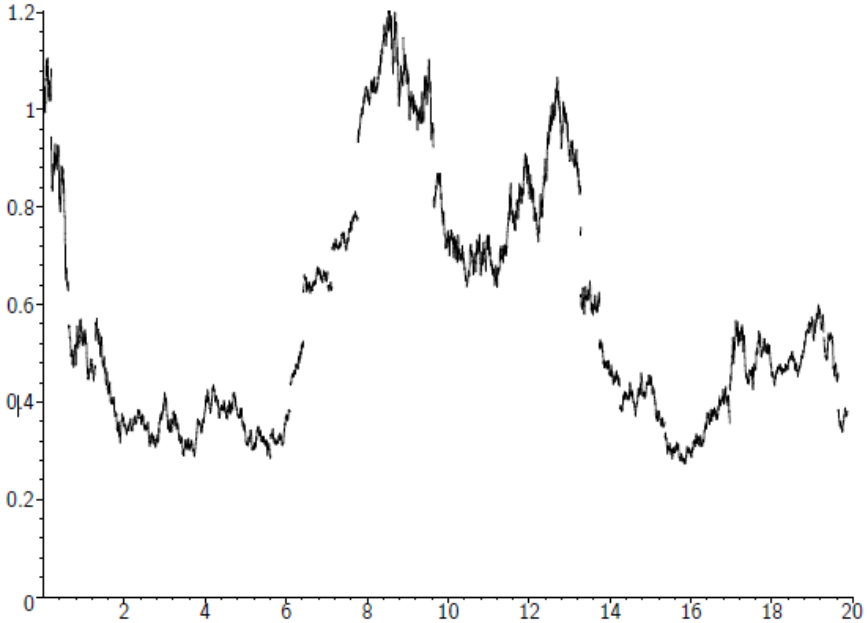

Figure: Path for the risky asset.

A simple example

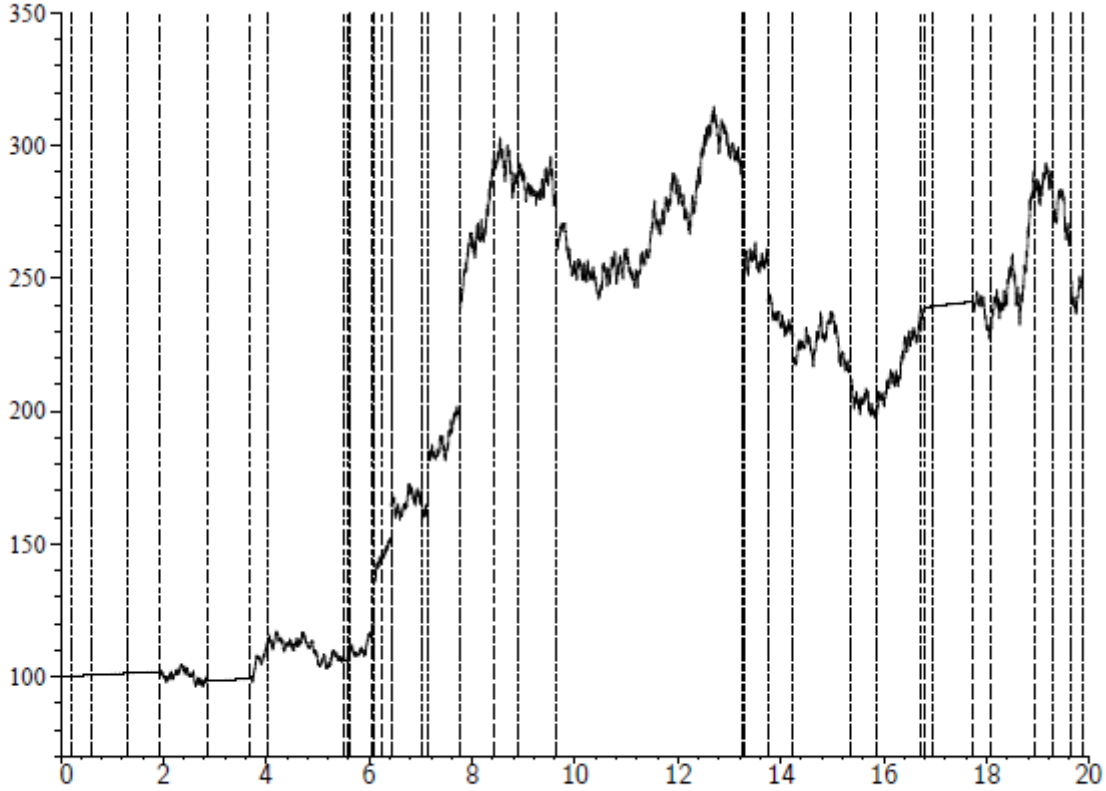

Figure: Sample path of the wealth process associated with the optimal portfolio for the consumption investment problem.

A simple example

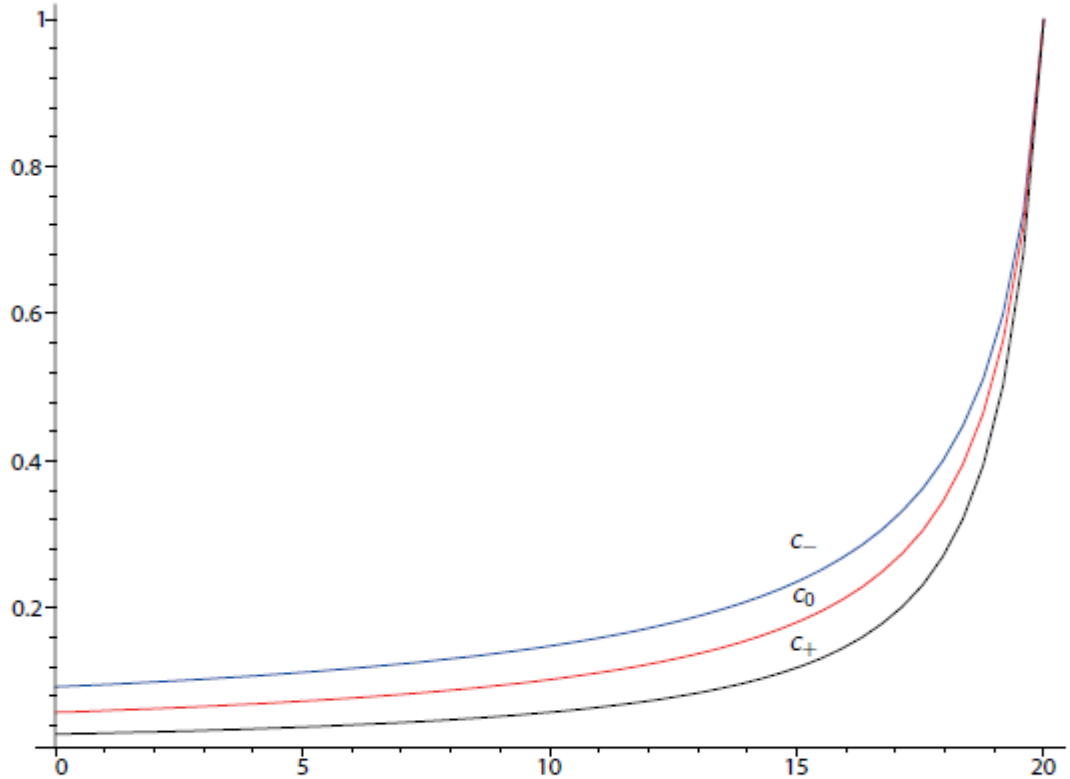

Figure: Plot of proportion of wealth consumed. The x axis represents the time, and the y axis the consumption rate c.

A simple example

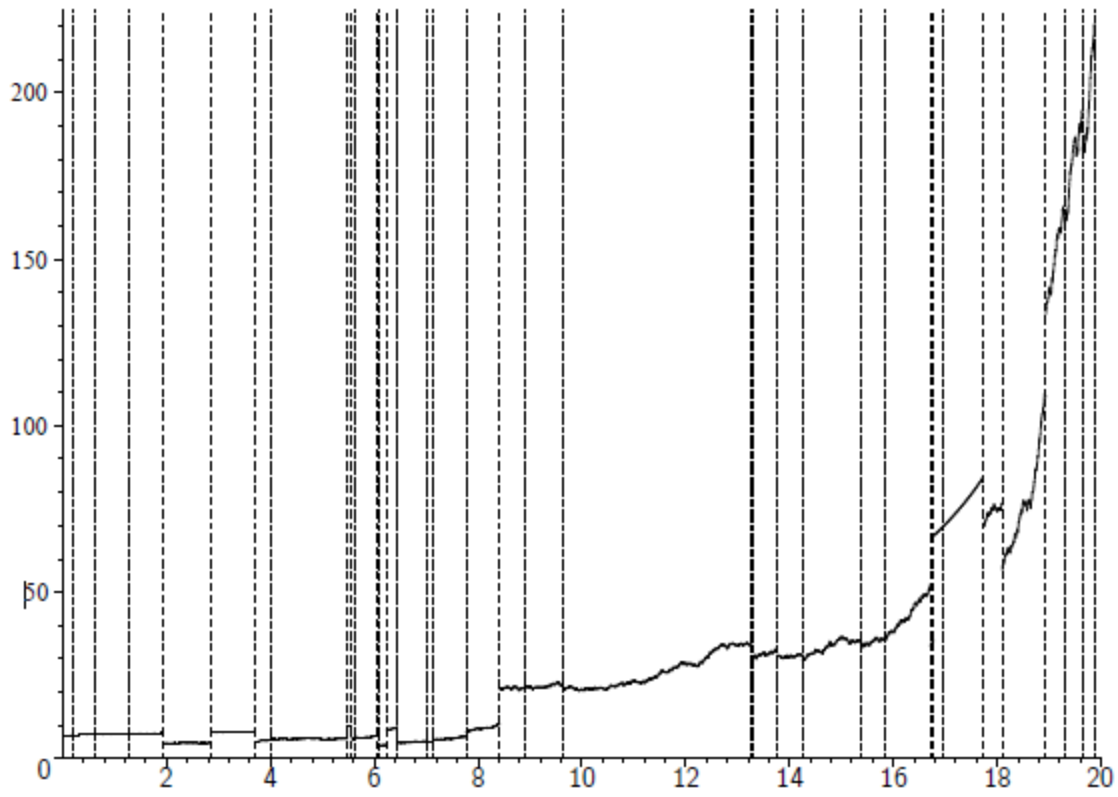

Figure: Optimal consumption path for the consumption investment problem.

(c) ***Analysis and Closed-Form Solutions:***

Emel Savku, G.-W. Weber

Maximum Principle for a Delayed Markov-switching jump-diffusion

Delay – Maximum Principle for a Markov Regime-Switching Jump-Diffusion Model

- We consider a stochastic optimal control problem where the system is given by a stochastic differential delay equation with jumps (SDDEJ) and Markov regime-switches.
- The stochastic maximum principle states that any optimal control satisfies a system of forward-backward SDDEs, called the *optimality system*, and maximizes a functional, called the *Hamiltonian*.
- The converse is also true, giving the sufficient maximum principle.

Delay – **Maximum Principle** for a Markov Regime-Switching Jump-Diffusion Model

By regime-switches, we can capture:

- The changes in the behaviors of financial markets, e.g., a shift from a volatile period to a calm period.
- The movements of financial markets between different states, e.g., growth, recession, crisis and bubble.
  - Discrete shifts from one regime to another may be activated by a change in economic policy, e.g., a shift in a monetary or an exchange rate policy or by a major event, e.g., the bankruptcy of Lehman Brothers in September 2008 or the 1973 oil crisis.

Delay – **Maximum Principle** for a Markov Regime-Switching Jump-Diffusion Model

- The delays in the dynamics can represent memory or inertia in the financial system.
- In the real world, investors tend to look at the historic performance of the risky asset.
  - If the price increases a lot, more investors tend to invest more money in this asset.
  - If the price decreases a lot, more investors tend to sell the asset and invest in other assets.

Delay – *Set up for Markov Switches*

- Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a complete filtred probability space where $\mathbb{F} = \{\mathscr{F}(t) : t \in [0,T]\}$ is a right continuous, $\mathbb{P}$-completed filtration.
- Let $\{\alpha(t) : t \in [0,T]\}$ be a homogeneous, irreducible continuous-time observable Markov chain with finite state space $S := \{e_1, e_2, ..., e_D\}$, where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$.
- Let the jth component of e_i be the Kronecker delta δ_{ij} for each $i,j = 1,2,...,D$.
- Let $\Lambda := [\lambda_{ij}]_{i,j=1,2,...,D}$ be the generator of the chain under $\mathbb{P}$, where for each $i,j = 1,2,...,D$, λ_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t.
- Let $\lambda_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{D} \lambda_{ij} = 0$.
- Let Brownian motion $W(\cdot)$, the Markov chain $\alpha(\cdot)$, Poisson random measure $N(\cdot,\cdot)$ be independent and adapted to $\mathbb{F}$.

Delay – *Set up for Markov Switches*

Elliot, Aggoun and Moore (1994) obtained the following semimartingale dynamics for the chain α:

$$\alpha(t) = \alpha(0) + \int_0^t \Lambda^T \alpha(u) du + M(t),$$

where $M(t)$ is a an $\mathbb{R}^D$-valued $(\mathbb{F}, \mathbb{P})$-martingale.
Let us define:

$$\begin{aligned} J^{ij}(t) &:= \sum_{0<s\leq t} \langle \alpha(s-), e_i \rangle \langle \alpha(s), e_j \rangle \\ &= \lambda_{ij} \int_0^t \langle \alpha(s-), e_i \rangle ds + \int_0^t \langle \alpha(s-), e_i \rangle \langle dM(s), e_j \rangle . \end{aligned}$$

We represent:

$$m_{ij}(t) := \int_0^t \langle \alpha(s-), e_i \rangle \langle dM(s), e_j \rangle ,$$

which is an $(\mathbb{F}, \mathbb{P})$-martingale.

Delay – *Set up for Markov Switches*

For each fixed $j = 1, 2, \ldots, D$, let Φ_j be the number of jumps into state e_j up to time t. Then,

$$\begin{aligned}\Phi_j(t) &:= \sum_{i=1, i\neq j}^{D} J^{ij}(t) \\ &= \sum_{i=1, i\neq j}^{D} \lambda_{ij} \int_0^t \langle \alpha(s-), e_i \rangle \, ds + \tilde{\Phi}_j(t),\end{aligned}$$

where $\tilde{\Phi}_j(t) := \sum_{i=1, i\neq j}^{D} m_{ij}(t)$ and $\lambda_j := \sum_{i=1, i\neq j}^{D} \lambda_{ij} \int_0^t \langle \alpha(s-), e_i \rangle \, ds$.

Hence, $\tilde{\Phi}_j(t) = \Phi_j(t) - \lambda_j(t)$ is an $(\mathbb{F}, \mathbb{P})$-martingale.

Delay – *Markov-Switching Jump-Diffusion*

Let us represent our model:

$$\begin{cases} dX(t) = b(t,X(t),Y(t),A(t),\alpha(t),u(t))dt \\ \qquad +\sigma(t,X(t),Y(t),A(t),\alpha(t),u(t))dW(t) \\ \qquad +\int_{\mathbb{R}_0}\eta(t,X(t),Y(t),A(t),\alpha(t),u(t),z)\tilde{N}(dt,dz) \\ \qquad +\gamma(t,X(t),Y(t),A(t),\alpha(t),u(t))d\tilde{\Phi}(t), \qquad t\in[0,T] \\ X(t) = x_0(t), \qquad t\in[-\delta,0] \text{ and } x_0\in C([-\delta,0];\mathbb{R}), \end{cases} \tag{1}$$

where

$$Y(t) = X(t-\delta) \text{ and } A(t) = \int_{t-\delta}^{t} e^{-\rho(t-r)}X(r)dr \text{ for } t\in[0,T].$$

$\delta > 0$, $\rho \geq 0$ and $T > 0$ are given constants and b, σ, η and γ are $\mathscr{F}_t$-measurable functions for all t.
Here, $\tilde{\Phi}(t) := (\tilde{\Phi}_1(t),\tilde{\Phi}_2(t),\ldots,\tilde{\Phi}_D(t))^T$ be the Markov jump-martingales associated with the chain α.

Delay – *Markov-Switching Jump-Diffusion*

Let U be a non-empty, closed, convex subset of $\mathbb{R}$. An admissible control is a U-valued, $\mathscr{F}_t$-measurable, *càdlàg* process $u(t)$, $t \in [0,T]$, such that the SDDE (1) has a unique solution and

$$E\left[\int_0^T |u(t)|^2 dt\right] < \infty.$$

We denote by $\mathscr{A}$ the set of all admissible controls.
Let us define the performance criterion (objective functional) as follows:

$$J(u) = E\left[\int_0^T f(t,X(t),Y(t),A(t),\alpha(t),u(t))dt + g(X(T),\alpha(T))\right]$$

for all $u \in \mathscr{A}$, where $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times S \times U \to \mathbb{R}$ and $g : \mathbb{R} \times S \to \mathbb{R}$ are C^1-functions with respect to x, y, a, u such that for $x_i = x, y, a, u$,

Delay – *Markov-Switching Jump-Diffusion*

$$E\Bigg[\int_0^T \Bigg(|f(t,X(t),Y(t),A(t),\alpha(t),u(t))| + \left|\frac{\partial f}{\partial x_i}(t,X(t),Y(t),A(t),\alpha(t),u(t))\right|^2\Bigg)dt + |g(X(T),\alpha(T))| + |g_x(X(T),\alpha(T))|^2\Bigg] < \infty.$$

Our problem is to find an optimal control $\hat{u} \in \mathscr{A}$ such that

$$J(\hat{u}) = \sup_{u\in\mathscr{A}} J(u(\cdot)) = \sup_{u\in\mathscr{A}} E\left[\int_0^T f(t,X(t),Y(t),A(t),\alpha(t),u(t))dt + g(X(T),\alpha(T))\right].$$

Delay – *Markov-Switching Jump-Diffusion*

We define the Hamiltonian,

$$H(t,x,y,a,e_i,u,p,q,r,s) = f(t,x,y,a,e_i,u) + b(t,x,y,a,e_i,u)p$$
$$+\sigma(t,x,y,a,e_i,u)q + \int_{\mathbb{R}_0} \eta(t,x,y,a,e_i,u,z)r(t,z)\nu(dz)$$
$$+\sum_{j=1}^{D} \gamma^j(t,x,y,a,e_i,u)\omega^j(t)\lambda_{ij}.$$

The adjoint equations corresponding to u and $X(t) := X^{(u)}(t)$ for the unknown, adapted processes $\{p(t)|t \in [0,T]\}$, $\{q(t)|t \in [0,T]\}$, $\{r(t,z)|t \in [0,T], z \in \mathbb{R}_0\}$, and $\{\omega(t)|t \in [0,T]\}$, where $p(t) \in \mathbb{R}$, $q(t) \in \mathbb{R}$, $r(t,z) \in \mathscr{R}$, and $\omega(t) \in \mathbb{R}^D$ are given by the following Anticipated Backward Stochastic Differential Equation (ABSDE):

Delay – *Markov-Switching Jump-Diffusion*

$$\begin{cases} dp(t) = E[\mu(t)|\mathscr{F}_t]dt + q(t)dW(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz) + \omega(t)d\tilde{\Phi}(t), \\ p(T) = g_x(X(T),\alpha(T)), \end{cases} \tag{2}$$

where

$$\begin{cases} \mu(t) = -\frac{\partial H}{\partial x}(t,X(t),Y(t),A(t),\alpha(t),u(t),p(t),q(t),r(t,\cdot),\omega(t)) \\ \qquad -\frac{\partial H}{\partial y}(t+\delta,X(t+\delta),Y(t+\delta),A(t+\delta),\alpha(t+\delta), \\ \qquad u(t+\delta),p(t+\delta),q(t+\delta),r(t+\delta,\cdot),\omega(t+\delta))\chi_{[0,T-\delta]}(t) \\ \qquad -e^{\rho t}(\int_t^{t+\delta} \frac{\partial H}{\partial a}(s,X(s),Y(s),A(s),\alpha(s),u(s),p(s),q(s), \\ \qquad r(s,\cdot),\omega(s))e^{-\rho s}\chi_{[0,T]}(s)ds). \end{cases} \tag{3}$$

Delay – *ABSDEs*

We will consider a new form for the BSDEs as follows:

$$-dY(t) = f(t, Y(t), Z(t), Q(t), V(t), Y(t+\delta_1(t)), Z(t+\delta_2(t)), Q(t+\delta_3(t)),$$
$$V(t+\delta_4(t)))dt - Z(t)dW(t) - \int_{\mathbb{R}_0} Q(t,z)\tilde{N}(dt,dz) - V(t)d\tilde{\Phi}(t),\ t \in [0,T];$$
$$Y(t) = \xi(t),\ Z(t) = \psi(t),\ Q(t) = \zeta(t) \text{ and } V(t) = \vartheta(t), \qquad t \in [T, T+K].$$

Let $\delta_i(\cdot)$, $i = 1,2,3,4$, be $\mathbb{R}^+$-valued continuous functions on $[0,T]$ such that:

(i) There exists a constant $K \geq 0$ such that for all $s \in [0,T]$ and $i = 1,2,3,4,\ \ s + \delta_i(s) \leq T + K$.

(ii) There exists a constant $L \geq 0$ such that for all $t \in [0,T]$ and some non-negative integrable function $g(\cdot)$,

$$\int_t^T g(s+\delta_i(s))ds \leq L \int_t^{T+K} g(s)ds \qquad \text{for} \qquad i = 1,2,3,4.$$

Delay – *ABSDEs*

Let us assume:

(H1) There exists a constant $C > 0$ such that for all $s \in [0,T]$, $y, y', z, z' \in \mathbb{R}$, $q, q' \in L^2(\mathscr{B}_0;\mathbb{R})$, $v, v' \in L^2(\mathscr{B}_S;\mathbb{R}^D)$, $\xi, \xi', \psi, \psi' \in L^2_{\mathbb{F}}(s, T+K;\mathbb{R})$, $\zeta, \zeta' \in \mathscr{H}^2_{\mathbb{F}}(s, T+K;\mathbb{R})$, $\vartheta, \vartheta' \in \mathscr{M}^2_{\mathbb{F}}(s, T+K;\mathbb{R}^D)$ and $r, r^*, \hat{r}, \tilde{r} \in [s, T+K]$, we have

$$
\begin{aligned}
&|f(s,y,z,q,v,\xi(r),\psi(r^*),\zeta(\hat{r}),\vartheta(\tilde{r})) \\
&\qquad -f(s,y',z',q',v',\xi'(r),\psi'(r^*),\zeta'(\hat{r}),\vartheta'(\tilde{r}))| \\
&\qquad \leq C\Big(|y-y'| + |z-z'| + \|q-q'\|_J + \|v-v'\|_S + E^{\mathscr{F}_s}\Big[|\xi(r)-\xi'(r)| \\
&+ |\psi(r^*)-\psi'(r^*)| + \|\zeta(\hat{r})-\zeta'(\hat{r})\|_J + \|\vartheta(\tilde{r})-\vartheta'(\tilde{r})\|_S\Big]\Big).
\end{aligned}
$$

(H2) $E\left[\int_0^T |f(s,0,0,0,0,0,0,0,0)|^2\,dt\right] < \infty.$

Delay – *ABSDEs*: *Existence - Uniqueness Theorem*

Theorem: Suppose f satisfies $(H1)$ and $(H2)$ and for $i = 1,2,3,4$, δ_i satisfies (i) and (ii). Then for any given terminal condition $\xi(\cdot) \in S^2_{\mathbb{F}}(T, T+K; \mathbb{R})$, $\psi(\cdot) \in L^2_{\mathbb{F}}(T, T+K; \mathbb{R})$, $\zeta(\cdot) \in \mathscr{H}^2_{\mathbb{F}}(T, T+K; \mathbb{R})$ and $\vartheta(\cdot) \in \mathscr{M}^2_{\mathbb{F}}(T, T+K; \mathbb{R}^D)$, the ABSDE has a unique solution, i.e., there exists a unique 4-tuple of $\mathscr{F}_t$-adapted processes $(Y, Z, Q, V) \in S^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times L^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathscr{H}^2_{\mathbb{F}}(0, T+K; \mathbb{R}) \times \mathscr{M}^2_{\mathbb{F}}(0, T+K; \mathbb{R}^D)$ satisfying the ABSDE.

Delay – *Necessary Condition for Optimality*

- Let $\hat{u}$ be an optimal control process and β "small" be such that $\hat{u}+\beta = v \in \mathscr{A}$. Since U is a convex set, for any $v \in \mathscr{A}$ nearby the perturbed control process $u^s = \hat{u} + s(v-\hat{u})$, $0 < s < 1$, is also in $\mathscr{A}$.
- The directional derivative of the performance criterion $J(\cdot)$ at $\hat{u}$ in the direction of $v-\hat{u}$ is given by:

$$\frac{d}{ds}J(\hat{u}+s(v-\hat{u}))|_{s=0} := \lim_{s\to 0^+} \frac{J(\hat{u}+s(v-\hat{u})) - J(\hat{u})}{s}.$$

- Since $\hat{u}$ is an optimal control, then a necessary condition for optimality is :

$$\frac{d}{ds}J(\hat{u}+s(v-\hat{u}))|_{s=0} \leq 0.$$

- Let the derivative process $\xi(t) := \frac{d}{ds}X^{u+s\beta}(t)\,|_{s=0}$ exist.

Delay – *Necessary Maximum Principle*

Theorem: Let $\hat{u} \in \mathscr{A}$ be an optimal control with the corresponding trajectory $(\hat{X}(t), \hat{Y}(t), \hat{A}(t))$ and $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z), \hat{\omega}(t))$ be the unique solution of the corresponding adjoint equation. Assume that,

$$E\Bigg[\int_0^T \hat{p}^2(t)\Bigg\{(\frac{\partial\hat{\sigma}}{\partial x})^2(t)\hat{\xi}^2(t) + (\frac{\partial\hat{\sigma}}{\partial y})^2(t)\hat{\xi}^2(t-\delta) + (\frac{\partial\hat{\sigma}}{\partial a})^2(t)$$
$$\left(\int_{t-\delta}^t e^{\rho(t-r)}\hat{\xi}(r)dr\right)^2 + (\frac{\partial\hat{\sigma}}{\partial u})^2(t)\beta^2(t) + \int_{\mathbb{R}_0}\Bigg\{(\frac{\partial\hat{\eta}}{\partial x})^2(t,z)\hat{\xi}^2(t)$$
$$+ (\frac{\partial\hat{\eta}}{\partial y})^2(t,z)\hat{\xi}^2(t-\delta) + (\frac{\partial\hat{\eta}}{\partial a})^2(t,z)\left(\int_{t-\delta}^t e^{\rho(t-r)}\hat{\xi}(r)dr\right)^2$$
$$+ (\frac{\partial\hat{\eta}}{\partial u})^2(t,z)\beta^2(t)\Bigg\}\nu(dz) + \sum_{j=1}^D\Bigg\{(\frac{\partial\hat{\gamma}^j}{\partial x})^2(t)\hat{\xi}^2(t) + (\frac{\partial\hat{\gamma}^j}{\partial y})^2(t)\hat{\xi}^2(t-\delta)$$
$$+ (\frac{\partial\hat{\gamma}^j}{\partial a})^2(t)\left(\int_{t-\delta}^t e^{\rho(t-r)}\hat{\xi}(r)dr\right)^2 + (\frac{\partial\hat{\gamma}^j}{\partial u})^2(t)\beta^2(t)\Bigg\}\lambda_j(t)\Bigg\}\,dt\Bigg] < \infty$$

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Delay – *Necessary Maximum Principle*

and

$$E\left[\int_0^T (\hat{\xi})^2(t)\left\{(\hat{q})^2(t)+\int_{\mathbb{R}_0}(\hat{r})^2(t,z)\nu(dz)+\sum_{j=1}^{D}(\hat{\omega}^j)^2(t)\lambda_j(t)\right\}dt\right]<\infty.$$

Then, for any $v \in U$ we have

$$\frac{\partial H}{\partial u}(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\alpha(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot),\hat{\omega}(t))(v-\hat{u}(t))\leq 0$$

$dt-a.e,\ \mathbb{P}-a.s..$

Delay – *Sufficient Maximum Principle*

Theorem: Let $\hat{u} \in \mathscr{A}$ with corresponding state processes $\hat{X}(t), \hat{Y}(t), \hat{A}(t)$ and adjoint processes $\hat{p}(t), \hat{q}(t), \hat{r}(t,z), \hat{\omega}(t)$, assumed to satisfy the ABSDE (2)-(3). Suppose that the following assertions hold:

$$E\Big[\int_0^T \hat{p}(t)^2((\sigma(t)-\hat{\sigma}(t))^2 + \int_{\mathbb{R}_0}(\eta(t,z)-\hat{\eta}(t,z))^2\nu(dz) + \sum_{j=1}^{D}(\gamma^j(t)-\hat{\gamma}^j(t))^2\lambda_j(t))dt\Big] < \infty$$

and

$$E\left[\int_0^T (X(t)-\hat{X}(t))^2\left\{\hat{q}^2(t) + \int_{\mathbb{R}_0}\hat{r}^2(t,z)\nu(dz) + \sum_{j=1}^{D}(\hat{\omega}^j)^2(t)\lambda_j(t)\right\}dt\right] < \infty$$

Delay – *Sufficient Maximum Principle*

Furthermore, we assume that the following conditions hold:

1. For almost all $t \in [0,T]$,

$$H(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\alpha(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot),\hat{\omega}(t))$$
$$= \max_{u\in U} H(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\alpha(t),\ u,\ \ \hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot),\hat{\omega}(t)).$$

2. H is a concave function of x,y,a,u for all $(t,e_i) \in [0,T]\times S$.
3. $g(x,e_i)$ is a concave function of x for each $e_i \in S$.

Then $\hat{u}(t)$ is an optimal control process and $\hat{X}(t),\hat{Y}(t),\hat{A}(t)$ are the corresponding controlled state processes.

Delay – *Partial Information*

- Let $\mathscr{E}_t \subseteq \mathscr{F}_t$, $t \in [0,T]$, be a given subfiltration of $\{\mathscr{F}_t\}_{t\in[0,T]}$ that represents the information available to the controller who decides the value of $u(t)$ at time t.
  For example, $\mathscr{E}_t = \mathscr{F}_{(t-c)^+}$ for some given $c > 0$.
- Let $U \subset \mathbb{R}$ be a given set of admissible control values $u(t)$, $t \in [0,T]$, and $\mathscr{A}_{\mathscr{E}}$ be a given family of admissible control processes included in the set of *càdlàg*, $\mathscr{E}$-adapted, U-valued processes such that the defined SHDDEJ, equation (1), has unique solution.

Delay – *Partial Information*

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Appendix

Delay – *Example*

We represent the model for an application to optimal consumption from a cash flow with delay for a Markov regime-switching jump-diffusion market:

- Let $b(t,\alpha(t))$, $\sigma(t,\alpha(t))$, $\eta(t,\alpha(t))$ and $\gamma(t,\alpha(t))$ be given bounded adapted processes.
- Let the consumption rate at any time $t \in [0,T]$, $c(t) \geq 0$, be a *càdlàg* adapted process.
- Then the corresponding net cash flow $X(t) = X^c(t)$ is:

$$\begin{cases} dX(t) = (Y(t)b(t,\alpha(t)) - c(t))dt + Y(t)\sigma(t,\alpha(t))dW(t) \\ \qquad +Y(t)\int_{\mathbb{R}_0}\eta(t,\alpha(t),z)\tilde{N}(dt,dz) \\ \qquad +Y(t)\gamma(t,\alpha(t))d\tilde{\Phi}(t), \qquad t \in [0,T]; \\ X(t) = x_0(t),\ t \in [-\delta,0] \text{ and } x_0 \in C([-\delta,0];\mathbb{R}). \end{cases} \tag{4}$$

Delay – *Optimal Consumption from a cash flow*

We represent the model for an application to optimal consumption from a cash flow with delay for a Markov regime-switching jump-diffusion market:

- Let $b(t,\alpha(t))$, $\sigma(t,\alpha(t))$, $\eta(t,\alpha(t))$ and $\gamma(t,\alpha(t))$ be given bounded adapted processes.
- Let the consumption rate at any time $t \in [0,T]$, $c(t) \geq 0$, be a *càdlàg* adapted process.
- Then the corresponding net cash flow $X(t) = X^c(t)$ is:

$$\begin{cases} dX(t) = (Y(t)b(t,\alpha(t)) - c(t))dt + Y(t)\sigma(t,\alpha(t))dW(t) \\ \qquad +Y(t)\int_{\mathbb{R}_0}\eta(t,\alpha(t),z)\tilde{N}(dt,dz) \\ \qquad +Y(t)\gamma(t,\alpha(t))d\tilde{\Phi}(t), \qquad t \in [0,T]; \\ X(t) = x_0(t),\ t \in [-\delta,0] \text{ and } x_0 \in C([-\delta,0];\mathbb{R}). \end{cases} \tag{4}$$

Delay – *Optimal Consumption from a cash flow*

Let $U(t,c,e_i)$, $i = 1,2,\ldots,D$, be a given stochastic utility function such that

- $t \mapsto U(t,c,e_i)$ is $\mathscr{F}_t$-adapted for each $c \geq 0$ and $e_i \in S$,
- $c \mapsto U(t,c,e_i)$ is C^1, $\frac{\partial U}{\partial c}(t,c,e_i) > 0$ for each $e_i \in S$,
- $c \mapsto \frac{\partial U}{\partial c}(t,c,e_i)$ is strictly decreasing for each $e_i \in S$,
- $\lim_{c\to\infty} \frac{\partial U}{\partial c}(t,c,e_i) = 0$ for all $t \in [0,T]$ and $e_i \in S$.
- Let $v_0(t,\alpha(t)) := \frac{\partial U}{\partial c}(t,0,\alpha(t))$ and define

$$I(t,v,\alpha(t)) := \begin{cases} 0, & \text{if } v \geq v_0(t,\alpha(t)) \\ \left(\frac{\partial U}{\partial c}(t,\cdot,\alpha(t))\right)^{-1}(v), & \text{if } 0 \leq v < v_0(t,\alpha(t)). \end{cases}$$

Delay – *Optimal Consumption from a cash flow*

Our problem is to find the consumption rate $\hat{c}(t)$ such that

$$J(\hat{c}) = \sup_{c \in \mathscr{A}} E\left[\int_0^T U(t, c(t), \alpha(t))dt + g(X(T), \alpha(T))\right]$$

In this case, the Hamiltonian gets the form:

$$H(t,x,y,a,e_i,c,p,q,r(\cdot),\omega) = U(t,c,e_i) + (b(t,e_i)y - c)p + y\sigma(t,e_i)q$$
$$+ y\int_{\mathbb{R}_0} \eta(t,e_i,z)r(t,z)\nu(dz) + y\sum_{j=1}^{D} \gamma^j(t,e_i)\omega^j(t)\lambda_{ij}.$$

Delay – *Optimal Consumption from a cash flow*

Hence, the corresponding adjoint equations are defined by:

$$
\begin{aligned}
dp(t) = -E[&(b(t+\delta,\alpha(t+\delta))p(t+\delta)+\sigma(t+\delta,\alpha(t+\delta))q(t+\delta)\\
&+\int_{\mathbb{R}_0}\eta(t+\delta,\alpha(t+\delta),z)r(t+\delta,z)\nu(dz)\\
&+\sum_{j=1}^{D}\gamma^j(t,\alpha(t+\delta))\omega^j(t)\lambda_j(t))\chi_{[0,T-\delta]}(t)|\mathscr{F}_t]dt\\
&+q(t)dW(t)+\int_{\mathbb{R}_0}r(t,z)\tilde{N}(dt,dz)+\omega(t)\tilde{\Phi}(t),\ t\in[0,T];
\end{aligned}
$$

$p(T) = g_x(X(T),\alpha(T))$.

Here we observe that maximizing H w.r.t. c gives

$$\frac{\partial U}{\partial c}(t,\hat{c}(t),\alpha(t)) = p(t).$$

Delay – *Optimal Consumption from a cash flow*

Proposition: Let $p(t), q(t), r(t,z), \omega(t)$ be the solution of the corresponding ABSDE and suppose that $0 \leq p(t) \leq v_0(t, \alpha(t))$ holds for all $t \in [0,T]$.
Then the optimal consumption rate $\hat{c}(t)$ and the corresponding optimal terminal wealth $X(t)$ are given implicitly by the coupled equations

$$\hat{c}(t) = I(t, p(t), \alpha(t))$$

and the equation (4), respectively.

Delay – *Optimal Consumption from a cash flow*

Let us assume that $b(t,\alpha(t)) = b(t)$ is deterministic and $g(X(t),\alpha(t)) = kX(t),\ k > 0$.
By Martingale Representation Theorem for regime-switching jump-diffusions, Crépéy and Matoussi [2008], we can choose $q = r = \omega = 0$.
Then the ABSDE (2)-(3) becomes:

$$\begin{cases} dp(t) = -b(t+\delta)p(t+\delta)\chi_{[0,T-\delta]}(t)dt, & t < T, \\ p(t) = k, \qquad t \in [T-\delta, T]. \end{cases}$$

Delay – *Optimal Consumption from a cash flow*

To solve this, we introduce

$$h(t) = p(T-t), \qquad t \in [0,T]$$

Then, we obtain the DDEs:

$$\begin{aligned} dh(t) &= b(T-t+\delta)h(t-\delta)dt, \qquad t \in [\delta, T], \\ h(t) &= k, \qquad t \in [0,\delta]. \end{aligned} \tag{5}$$

Hence, we can determine $h(t)$ inductively on each interval as follows:

$$h(t) = h(j\delta) + \int_{j\delta}^{t} h'(s)ds = h(j\delta) + \int_{j\delta}^{t} b(T-t+\delta)h(s-\delta)ds \tag{6}$$

for $t \in [j\delta, (j+1)\delta]$.

Delay – *Optimal Consumption from a cash flow*

Proposition: Assume that for each $e_i,\ i=1,2,\ldots,D$, and $t\in[0,T]$, $b(t,e_i)>0$ is deterministic, $U(t,c,e_i)=\phi(t,e_i)\ln(1+c)$, where $\phi(t,e_i)>0$ and $g(x,e_i)=kx,\ k>0$. Let for each $e_i,\ i=1,2,\ldots,D$, and $t\in[0,T]$, be $h_\delta(T-t)<\phi(t,e_i)$. Then,

$$\hat{c}(t)=\frac{\phi(t,e_i)}{h_\delta(T-t)}-1,$$

where $h_\delta(\cdot)=h(\cdot)$ is determined by equations (5)-(6).

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Thank you very much for your attention!

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Delay – *Partial Information*

- Let $\mathscr{E}_t \subseteq \mathscr{F}_t$, $t \in [0,T]$, be a given subfiltration of $\{\mathscr{F}_t\}_{t\in[0,T]}$ that represents the information available to the controller who decides the value of $u(t)$ at time t.
  For example, $\mathscr{E}_t = \mathscr{F}_{(t-c)^+}$ for some given $c > 0$.
- Let $U \subset \mathbb{R}$ be a given set of admissible control values $u(t)$, $t \in [0,T]$, and $\mathscr{A}_{\mathscr{E}}$ be a given family of admissible control processes included in the set of *càdlàg*, $\mathscr{E}$-adapted, U-valued processes such that the defined SHDDEJ, equation (1), has unique solution.

Delay – *Partial Information*

- **A1** For all $u \in \mathscr{A}_{\mathscr{E}}$ and all bounded β there exists $\varepsilon > 0$ such that
$$u + s\beta \in \mathscr{A}_{\mathscr{E}}$$
for all $s \in (-\varepsilon, \varepsilon)$.
- **A2** For all $t_0 \in [0, T]$ and for all bounded $\mathscr{E}_{t_0}$-measurable random variables v, the control process $\beta(t)$, defined by
$$\beta(t) = v\chi[t_0, T](t), \; t \in [0, T],$$
belongs to $\mathscr{A}_{\mathscr{E}}$.
- **A3** For all bounded $\beta \in \mathscr{A}_{\mathscr{E}}$ the derivative process
$$\xi(t) := \frac{d}{ds} X^{u+s\beta}(t)\,|_{s=0}$$
exists.

Delay – *Necessary Maximum Principle under Partial Information*

Theorem: Let $\hat{u} \in \mathscr{A}_{\mathscr{E}}$ be an optimal control with the corresponding trajectory $(\hat{X}(t), \hat{Y}(t), \hat{A}(t))$ and $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z), \hat{\omega}(t))$ be the unique solution of the corresponding adjoint equation. Under some technical assumptions, the following conditions are equivalent.

(i) $\frac{d}{ds} J(\hat{u} + s\beta)|_{s=0} = 0$ for all $\beta \in \mathscr{A}_{\mathscr{E}}$.

(ii) $E\left[\frac{\partial H}{\partial u}(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t),\ u,\ \ \alpha(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot), \hat{\omega}(t)) | \mathscr{E}_t\right]_{u=\hat{u}(t)}$
$= 0$ a.s. for all $t \in [0,T]$.

Delay – *Sufficient Maximum Principle under Partial Information*

Theorem: Let $\hat{u} \in \mathscr{A}_{\mathscr{E}}$ with corresponding state processes $(\hat{X}(t), \hat{Y}(t), \hat{A}(t))$ and suppose there exists an adapted solution $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z), \hat{\omega}(t))$ of the corresponding adjoint equation such that for all $u \in \mathscr{A}_{\mathscr{E}}$ we have

$$E\Big[\int_0^T \hat{p}(t)^2((\sigma(t)-\hat{\sigma}(t))^2 + \int_{\mathbb{R}_0} (\eta(t,z)-\hat{\eta}(t,z))^2 \nu(dz) + \sum_{j=1}^{D} (\gamma^j(t)-\hat{\gamma}^j(t))^2 \lambda_j(t))dt\Big] < \infty$$

and

$$E\left[\int_0^T (X(t)-\hat{X}(t))^2 \left\{\hat{q}^2(t) + \int_{\mathbb{R}_0} \hat{r}^2(t,z)\nu(dz) + \sum_{j=1}^{D} (\hat{\omega}^j)^2(t)\lambda_j(t)\right\} dt\right] < \infty.$$

Delay – *Sufficient Maximum Principle under Partial Information*

Furthermore, we assume that the following conditions hold:

1. For almost all $t \in [0,T]$,

$$E[H(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\alpha(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot),\hat{\omega}(t))|\mathscr{E}_t]$$
$$= \max_{u\in U} E[H(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\alpha(t),\ u,\ ,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot),\hat{\omega}(t))|\mathscr{E}_t].$$

2. H is a concave function of x,y,a,u for all $(t,e_i) \in [0,T]\times S$.
3. $g(x,e_i)$ is a concave function of x for each $e_i \in S$.

Then $\hat{u}(t)$ is an optimal control process and $\hat{X}(t),\hat{Y}(t),\hat{A}(t)$ are the corresponding controlled state processes.

Bubbles, Jumps and Insiders

Notation:

$$(\Omega, \mathscr{F}, P), \quad (\Omega, P) = (\Omega_B \times \Omega_\eta, P_B \times P_\eta),$$

$$\eta(t) = \int_0^t \int_{\mathbb{R}} z\, \tilde{N}(dt, dz) \geq 0, \quad \tilde{N}(dt, dz) = (N - v_{\mathscr{F}})(dt, dz) = N(dt, dz) - v_{\mathscr{F}}(dz)dt,$$

$$\mathscr{F}_t^B := \sigma\left\{B(s) \mid s < t, t \in [0,T]\right\} \vee \mathscr{N}, \quad \mathscr{F}_t^{\tilde{N}} := \sigma\left\{\tilde{N}(\Delta) \mid \Delta \in \mathscr{B}(\mathbb{R}_0 \times (0,t)),\ t \in [0,T]\right\} \vee \mathscr{N},$$

$$\mathscr{F}_t = \mathscr{F}_t^B \otimes \mathscr{F}_t^{\tilde{N}},$$

$$\mathscr{F}_t \subseteq \mathscr{G}_t \subseteq \mathscr{F} \quad \forall t \in [0,T],$$

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$$(\Omega, \mathscr{F}, P), \quad (\Omega, P) = (\Omega_B \times \Omega_\eta, P_B \times P_\eta),$$

$$\eta(t) = \int_0^t \int_{\mathbb{R}} z\, \tilde{N}(dt, dz) \geq 0, \quad \tilde{N}(dt, dz) = (N - v_{\mathscr{F}})(dt, dz) = N(dt, dz) - v_{\mathscr{F}}(dz)dt,$$

pure jump Lévy process compensated Poisson random measure Lévy measure

$$\mathscr{F}_t^B := \sigma\left\{B(s) \mid s < t, t \in [0,T]\right\} \vee \mathscr{N}, \quad \mathscr{F}_t^{\tilde{N}} := \sigma\left\{\tilde{N}(\Delta) \mid \Delta \in \mathscr{B}(\mathbb{R}_0 \times (0,t)),\ t \in [0,T]\right\} \vee \mathscr{N},$$

$$\mathscr{F}_t = \mathscr{F}_t^B \otimes \mathscr{F}_t^{\tilde{N}},$$

$$\mathscr{F}_t \subseteq \mathscr{G}_t \subseteq \mathscr{F} \quad \forall t \in [0,T],$$

Bubble

Bubbles, Jumps and Insiders

$$\int_0^\infty \varphi(t,\omega)d^-B(t) = \lim_{\varepsilon\to 0}\int_0^\infty \varphi(t,\omega)\frac{B(t+\varepsilon)-B(t)}{\varepsilon}dt$$

Forward Integrals

$$\int_0^\infty \int_{\mathbb{R}_0} \varphi(t,\omega)\tilde{N}(d^-t,dz) = \lim_{n\to\infty}\int_0^\infty \int_{U_n} \varphi(t,z)\tilde{N}(dt,dz)$$

$$(U_n \subset \mathbb{R}_0 \text{ compact},\ U_n \subseteq U_{n+1}\ (n\in \mathrm{IN}),\ \nu_F\left(U_n\right)<\infty,\ \bigcup_{n=1}^\infty U_n = \mathbb{R}_0)$$

$$X(t) = X(0) + \int_0^t \alpha(s)ds + \int_0^t \beta(s)d^-B(s) + \int_0^\infty \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(d^-s,dz) \quad \forall t\in[0,T]$$

$$d^-X(t) = \alpha(t)dt + \beta(t)d^-B(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(d^-t,dz)$$

Forward Process

Bubbles, Jumps and Insiders

If $Y(t) = f(X(t))$:

Itô Formula for Forward Integrals

$$d^{-}Y(t) = \Big[f'(X(t))\alpha(t) + \frac{1}{2} f''(X(t))\alpha(t)\beta(t)^2 + \int_{\mathbb{R}_0} \Big\{ f(X(t^-) + \gamma(t,z))$$

$$- f(X(t^-)) - f'(X(t^-))\gamma(t,z) \Big\} \nu_{\mathcal{F}}(dz) \Big] dt + f'(X(t))\beta(t) d^{-}B(t)$$

$$+ \int_{\mathbb{R}_0} \Big(f(X(t^-) + \gamma(t,z)) - f'(X(t^-)) \Big) \tilde{N}(d^{-}t, dz).$$

Bubbles, Jumps and Insiders

$$dS_0(t) = r(t)S_0(t)dt,$$
$$S_0(0) = 1,$$

$$dS_1(t) = S_1(t^-)\left[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz)\right],$$
$$S_1(0) > 0,$$

$$d^-X^{(c,\pi)}(t) = X^{(c,\pi)}(t^-)\Big[\{r(t) + (\mu(t) - r(t))\pi(t)\}dt + \sigma(t)\pi(t)d^-B(t)$$
$$+\pi(t)\int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(d^-t,dz)\Big] - c(t)dt,$$
$$X^{(c,\pi)}(0) = \upsilon.$$

Bubbles, Jumps and Insiders

If $\lambda(t) = \dfrac{c(t)}{X^{(c,\pi)}(t)}$, then

$$d^- X^{(c,\pi)}(t) = X^{(c,\pi)}(t^-)\Big[\big\{r(t) - \lambda(t) + (\mu(t) - r(t))\pi(t)\big\}\,dt$$

$$+ \sigma(t)\pi(t)d^- B(t) + \pi(t)\int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(d^- t, dz)\Big],$$

$$J(\lambda^*, \pi^*) := \sup_{(\lambda,\pi)\in\mathcal{A}} \mathrm{IE}\left[\int_0^T e^{-\delta(t)} \ln\Big(\lambda(t) X^{(c,\pi)}(t)\Big)dt + Ke^{-\delta(T)} \ln X^{(c,\pi)}(T)\right].$$

Bubbles, Jumps and Insiders

Mixed Case

$\gamma(t,z) = z, \quad \sigma(t) \neq 0 \quad$ for almost all (t,z)

$\mathcal{G}'_t = \mathcal{F}_t \vee \sigma(B(T_0), \eta(T_0)), \quad T_0 > T$

Assumptions : (1) Uninformed agent has access to filtration $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{G}'_t \ (t \in [0,T])$,

(2) Lévy measure $\nu_{\mathcal{F}}$ is given by $\nu_{\mathcal{F}}(ds,dz) = \rho\delta_1(dz)ds \quad (\delta_1(dz)$: unit point mass at 1),

(3) $\eta(t)$ is given by $\eta(t) = Q(t) - \rho t \quad (Q$: Poisson process of intensity $\rho)$.

$$\pi_i^*(t) = \pi_h^*(t) + \frac{\varsigma(t)}{\sigma(t)} \qquad \pi_h^*(t) = \frac{\mu(t) - r(t)}{\sigma(t)^2} - \frac{\rho}{\sigma(t)^2} \qquad \lambda_i^*(t) = \lambda_h^*(t) = \frac{e^{-\delta(t)}}{\int_t^T e^{-\delta(s)}ds + Ke^{-\delta(T)}}$$

$$\varsigma(t) = \frac{1}{2\sigma(t)}\Big[-\mu(t) + r(t) + \rho + \sigma(t)\alpha(t) - \sigma(t)^2 + \sqrt{\left(\mu(t) - r(t) - \rho + \sigma(t)\alpha(t) + \sigma(t)^2\right) + 4\sigma(t)^2\theta(t)}\Big],$$

$$\alpha(t) = \frac{\mathrm{IE}\left[B(T_0) - B(t) \mid \mathcal{G}_t\right]_-}{T_0 - t} \qquad \theta(t) = \frac{\mathrm{IE}\left[Q(T_0) - Q(t) \mid \mathcal{G}_t\right]_-}{T_0 - t}$$

Bubbles, Jumps and Insiders

Mixed Case

$$
\begin{aligned}
J_i(c_{\lambda_i^*},\pi_i^*) = J_h(c_{\lambda_h^*},\pi_h^*) &+ \mathrm{IE}\left[\int_0^T e^{-\delta(t)}\int_0^t\left(-\frac{1}{2}\varsigma(s)^2+\varsigma(s)\alpha(s)\right)dsdt\right] \\
&+ \mathrm{IE}\left[\int_0^T e^{-\delta(t)}\int_0^t\int_{\mathbb{R}}\ln\left(1+\frac{z\varsigma(s)\sigma(s)}{\sigma(s)^2+\left(\mu(s)-r(s)-\rho\right)z}\right)\nu_{\varsigma}(ds,dz)dt\right] \\
&+ K\,\mathrm{IE}\left[e^{-\delta(T)}\int_0^T\left(-\frac{1}{2}\varsigma(s)^2+\varsigma(s)\alpha(s)\right)ds\right] \\
&+ K\,\mathrm{IE}\left[e^{-\delta(T)}\int_0^T\int_{\mathbb{R}}\ln\left(1+\frac{z\varsigma(s)\sigma(s)}{\sigma(s)^2+\left(\mu(s)-r(s)-\rho\right)z}\right)\nu_{\varsigma}(ds,dz)\right]
\end{aligned}
$$

D. David and Y. Yolcu Okur, 2009

Dynamic Programming

- For any $(s, y, i) \in [0, T) \times \mathbb{R}^N \times S$, consider:
  - State equation

$$
\begin{aligned}
\mathrm{d}X(t) &= f\left(t, X(t_-), \alpha(t_-), u(t_-)\right)\mathrm{d}t + g\left(t, X(t_-), \alpha(t_-), u(t_-)\right)\mathrm{d}W(t) \\
&\quad + \int_{\mathbb{R}_0^K} h\left(t, X(t_-), \alpha(t_-), u(t_-), z\right)\tilde{J}(\mathrm{d}t, \mathrm{d}z)\ , \quad t \in [s, T]. \\
X(s) &= y\ , \quad \alpha(s) = i.
\end{aligned}
$$

  - Objective functional

$$
J(s, y, i; u(\cdot)) = E\left[\int_s^T L\left(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)\right)\mathrm{d}t \right.
$$

$$
\left. + \Psi\left(T, X_{s,y,i}(T; u(\cdot)), \alpha_{s,i}(T)\right)\right].
$$

Dynamic Programming

- For each $s \in [0, T)$, denote by $\mathcal{U}^w[s, T]$ the set of *weak admissible controls*:
  - 8-tuples of the form

$$\tilde{u} = (\Omega, \mathcal{F}, \mathbb{F}, P, W(\cdot), \alpha(\cdot), \eta(\cdot), u(\cdot)).$$

- Optimal control problem in dynamic programming form:
  - For any $(s, y, i) \in [0, T) \times \mathbb{R}^N \times S$, find $\bar{u}(\cdot) \in \mathcal{U}^w[s, T]$ such that

$$J(s, y, i; \bar{u}(\cdot)) = \sup_{u(\cdot) \in \mathcal{U}^w[s, T]} J(s, y, i; u(\cdot)).$$

- The *value function* $V : [0, T] \times \mathbb{R}^N \times S \to \mathbb{R}$ is well-defined by

$$\begin{cases} V(s, y, i) = \sup\limits_{u(\cdot) \in \mathcal{U}^w[s, T]} J(s, y, i; u(\cdot)) \\ V(T, y, i) = \psi(T, y, i) \end{cases} \qquad (s, y, i) \in [0, T) \times \mathbb{R}^N \times S.$$

Dynamic Programming Principle

Theorem (Bellman's Principle of Optimality)

Assume that conditions (A1)-(A2) hold. Then, for any $(s, y, i) \in [0, T) \times \mathbb{R}^N \times S$ *we have that*

$$V(s,y,i) = \sup_{u \in \mathcal{U}^w[s,T]} E\left[\int_s^{\hat{s}} L\left(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)\right) \mathrm{d}t \right.$$
$$\left. + V\left(\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s})\right) \right]$$

for all $\hat{s} \in [s, T]$.

Dynamic Programming Principle and HJB Equation

- The dynamic programming principle can be used to derive a HJB equation:
  - partial integro-differential equation whose "solution" is the value function of the optimal control problem.

Theorem (Hamilton-Jacobi-Bellman Equation)

Suppose that conditions (A1)-(A2) hold and that the value function V is such that $V(\cdot,\cdot,\alpha) \in C^{1,2}\left([0,T) \times \mathbb{R}^N; \mathbb{R}\right)$ for every $\alpha \in S$. Then, for each $\alpha \in S$, the value function $V(\cdot,\cdot,\alpha)$ defined on $[0,T) \times \mathbb{R}^N$ is the solution of the Hamilton-Jacobi-Bellman equation:

$$\begin{cases} V_t + \sup\limits_{u \in U} \mathcal{H}(t, X, \alpha, u, V, V_X, V_{XX}) = 0, \\ V(T, X, \alpha) = \Psi(T, X, \alpha)\,, \qquad (t, X, \alpha) \in [0,T) \times \mathbb{R}^N \times S. \end{cases}$$

Dynamic Programming Principle and HJB Equation

Theorem (Hamilton-Jacobi-Bellman Equation)

where the Hamiltonian function $\mathcal{H}(t, X, \alpha, u, V, V_X, V_{XX})$ *is given by*

$$\begin{aligned}\mathcal{H}(t, X, \alpha, u, V, V_X, V_{XX}) = L(t, X, \alpha, u) &+ \langle V_X(t, X, \alpha), f(t, X, \alpha, u)\rangle \\ &+ \frac{1}{2}\mathrm{tr}\left(g^T(t, X, \alpha, u)V_{XX}(t, X, \alpha)g(t, X, \alpha, u)\right) \\ &+ \sum_{j \in S: j \neq \alpha} q_{\alpha j}\left(V(t, X, j) - V(t, X, \alpha)\right) \\ &+ \sum_{k=1}^{K} \int_{\mathbb{R}_0^1} W_k(t, X, \alpha, u, V, V_X, z_k)\nu_k(\mathrm{d}z_k)\end{aligned}$$

and

$$\begin{aligned}W_k(t, X, \alpha, u, V, V_X, z_k) \quad = \quad & V(t, X + h^{(k)}(t, X, \alpha, u, z_k), \alpha) - V(t, X, \alpha) \\ & -\langle V_X(t, X, \alpha), h^{(k)}(t, X, \alpha, u, z_k)\rangle.\end{aligned}$$