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Transportation Interval Situations and Related Games

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Content

Introduction

- In many situations, producers and retailers are aiming to minimize their costs or maximizing their profits.
- Producers and retailers can form coalitions in order to obtain/save as much as possible. Constitutively, a **transportation situation** consists of two sets of **agents** called producers and retailers which produce/demand goods.
  The transport of the goods from the producers to the retailers has to be profitable.
- Therefore, the main objective is to transport the goods from the producers to the retailers at maximum profit (Aparicio et al. (2010)). Such a cooperation can occur in transportation situations (Sanchez Soriano et al. (2001, 2006)). However, when the agents involved agree on a **coalition**, the question of distributing the obtained benefit or costs among the agents arises.
- Cooperative game theory is widely used on interesting sharing cost/profit problems in the areas Operations Research such as connection, routing, scheduling, production and inventory, transportation situations (Borm et al. (2001)).

What Has Been Done

- Transportation games are examined in Sanchez-Sorianao et al. (2001).
- Our paper surveys the core of the transportation games and proves the nonemptiness of the core of transportation games.
- Moreover, they provide some results about the relationship between the core and dual optimal solutions of the transportation problem.
- Sanchez-Soriano (2003) introduces an ad hoc solution concept for transportation games called the pairwise Egalitarian solution.
- In the sequel, Sanchez-Sorianao et al. (2006) examines the relationship between the so-called pairwise solutions and the core of transportation games.
- Furthermore, they show that every core element of a transportation game is contained in a pairwise solution with a specific weight system.

What Has Been Done: Continued

- In the classical approach to the problem, the parameters are exactly known. In this case the problem is fully solved using the results of Sanchez-Sorianao et al. (2001). However, in real-life transportation situations, problem parameters are not known exactly. Agents considering cooperation can rather forecast the lower and upper bounds for the outcome of their cooperation. Thus, we have a transportation interval situation and to solve the related sharing benefit problems we need suitable sets of solutions.
- To handle transportation situations with interval data, the theory of cooperative interval games is suitable: Alparslan Gök et al. (2008),(2009a,b). The reader is referred to Branzei et al. (2010a,b), for a brief survey on cooperative solution concepts and for a guide for using interval solutions, when uncertainty about data is removed (Alparslan Gök et al., 2011).
- This work extends the analysis of two-sided transportation situations in Sanchez-Soriano (2006) , and their related cooperative games to a setting with interval data, i.e., the profit b_{ij} of goods j by producer i, the production p_i of goods of producer i, and the demand q_j of goods retailer j, in the transportation model now lie in intervals of real numbers obtained by forecasting their values from the aspect expert view.

Preleminaries

Definition 1: *A cooperative interval game is an ordered pair* $< N, v' >$, *where* $N = \{1, \dots, n\}$ *is the set of players, and* $v' : 2^N \to I(\mathbb{R})$ *is the characteristic function such that* $v'(\emptyset) = [0,0]$. *Here,* $I(\mathbb{R})$ *is the set of all nonempty, compact intervals in* $\mathbb{R}$. *For each* $S \in 2^N$, *the worth set (or worth interval)* $v'(S)$ *of the coalition* S *in the interval game* $< N, v' >$ *is of the form* $[\underline{v}'(S), \overline{v}'(S)]$, *where* $\underline{v}'(S)$ *is the minimal reward which coalition* S *could receive on its own and* $\overline{v}'(S)$ *is the maximal reward which coalition* S *could get.*

The family of all interval games with player set N is denoted by IG^N. We note that, if all the worth intervals are degenerate intervals, i.e., $\underline{v}'(S) = \overline{v}'(S)$ for each $S \in 2^N$, then the interval game $< N, v' >$ corresponds in a natural way to the classical cooperative game $< N, v >$, where $v(S) = \underline{v}'(S)$ for all $S \in 2^N$.

Preleminaries: Interval Calculus

Some classical cooperative games associated with an interval game $v' \in IG^N$ will play a key role, namely, the border games $< N, \underline{v'} >$, $< N, \overline{v'} >$ and the length game $< N, |v'| >$, where $|v'|(S) = \overline{v'}(S) - \underline{v'}(S)$ for each $S \in 2^N$. We note that $\overline{v'} = \underline{v'} + |v'|$.

Let $I, J \in I(\mathbb{R})$ with $I = [\underline{I}, \overline{I}]$, $J = [\underline{J}, \overline{J}]$, $|I| = \overline{I} - \underline{I}$ and $\alpha \in \mathbb{R}_+$. Then,

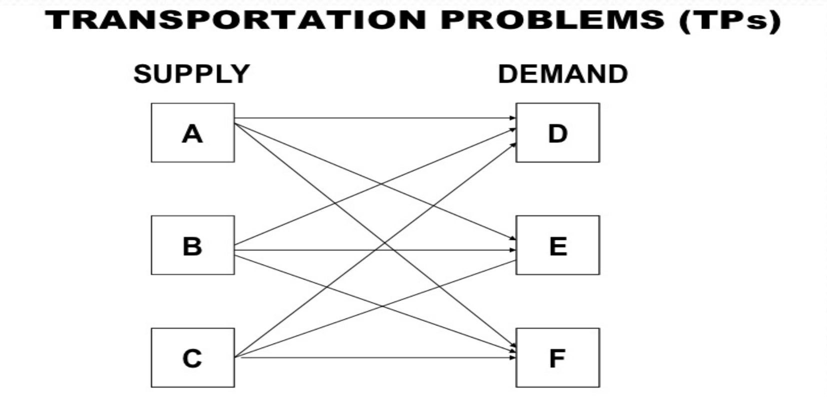
- $I + J = [\underline{I}, \overline{I}] + [\underline{J}, \overline{J}] = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$;
- $\alpha I = \alpha[\underline{I}, \overline{I}] = [\alpha\underline{I}, \alpha\overline{I}]$.

By (i) and (ii) we see that $I(\mathbb{R})$ has a cone structure.

- In this paper we also need a partial substraction operator. We define $I - J$, only if $|I| \geq |J|$, by $I - J =: [\underline{I}, \overline{I}] - [\underline{J}, \overline{J}] = [\underline{I} - \underline{J}, \overline{I} - \overline{J}]$. Let us note that $\underline{I} - \underline{J} \leq \overline{I} - \overline{J}$. We recall that I is weakly better than J, which we denote by $I \mu J$, if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$. Furthermore, we use the reverse notation $I' J$, if and only if $\underline{I} \leq \underline{J}$ and $\overline{I} \leq \overline{J}$. We say that I is better than J, which we denote by $I \succ J$, if and only if $I \succ= J$ and $I \neq J$.

Transportation Situations

In a *transportation situation* the set of players is partitioned into two disjoint subsets P and Q, containing n and m players, respectively. The members of P will be called origin players, whereas the members of Q will be the destination players. Each origin player $i \in P$ has a positive integer number of units of a certain indivisible good, p_i, and each destination player $j \in Q$ demands a positive integer number of units of this good, q_j. The shipping of one unit from origin player i to destination player j produces a nonnegative real profit b_{ij}.



Definition 2: *A transportation situation like this is characterized by a 5-tuple (P, Q, B, p, q), where B is the $n{\times}m$ matrix of profits, p is the n −dimensional vector of available units at the origins, and q is the m −dimensional vector of demands.*

Mathematical Modelling of Transportation Situations

Definition 3: *For every transportation situation* (P, Q, B, p, q) *and every coalition* $S \subset N := P \cup Q$, *with origin players* $S_P := S \cap P$ *and destination players* $S_Q := S \cap Q$, *and assuming that these sets are both non-empty, we can define the maximization problem by:*

$$\boldsymbol{T}(\boldsymbol{S}): \quad \textbf{maximize} \quad \sum_{i \in S_P} \sum_{j \in S_Q} b_{ij} x_{ij}$$

$$\textbf{such that} \quad \sum_{j \in S_Q} x_{ij} \le p_i, \qquad i \in S_P,$$

$$\sum_{j \in S_P} x_{ij} \le q_j, \qquad j \in S_Q,$$

$$x_{ij} \ge 0, \qquad (i, j) \in S_P \times S_Q.$$

Transportation Games

If we denote by $\vartheta(T(S))$ the optimal value of the problem $T(S)$, we can define a *TU-game* associated with every transportation situation (P, Q, B, p, q) in the following way:

- The set of players is $N = P \cup Q$;
- The characteristic function v is given by:

$$v(S) = \begin{cases} 0 & \text{, if } S = \emptyset \text{ or } S \text{ is contained in } P \text{ or in } Q, \\ \vartheta\big(T(S)\big) & \text{, in any other case.} \end{cases}$$

Now we give the definition of a transportation game.

Definition 4. *A transportation game is any TU-game $v \in G^N$ arising from a transportation situation (P, Q, B, p, q). Often, we identify a transportation situation (P, Q, B, p, q) with its associated transportation game v.*

Transportation Interval Situations

A *transportation interval situation* like this is characterized by a 5-tuple $(P, Q, B^{'}, p^{'}, q^{'})$, where $B^{'}$ is the $n\times m$ matrix of interval profits, $p^{'}$ is the $n-$dimensional vector of available interval units at the origins, and $q^{'}$ is the $m-$dimensional vector of interval demands.

For every transportation interval situation $(P, Q, B^{'}, p^{'}, q^{'})$ and every coalition $S \subset N := P \cup Q$, with origin players $S_P := S \cap P$ and destination players $S_Q := S \cap Q$, and assuming that these sets are both non-empty, we can define the maximization problem of the pessimistic scenario is expressed by:

$$
\begin{aligned}
\boldsymbol{T(S)}: \quad & \textbf{\textit{maximize}} && \sum_{i\in S_P}\sum_{j\in S_Q} \underline{b'_{ij}}\, x_{ij} \\
& \textbf{\textit{such that}} && \sum_{j\in S_Q} x_{ij} \le \underline{p'_i}, \quad i \in S_P, \\
& && \sum_{j\in S_P} x_{ij} \le \underline{q'_j}, \quad j \in S_Q, \\
& && x_{ij} \ge 0, \quad (i,j) \in S_P\times S_Q,
\end{aligned}
$$

and the maximization problem of the optimistic scenario is stated as:

$$
\begin{aligned}
\boldsymbol{T(S)}: \quad & \textbf{\textit{maximize}} && \sum_{i\in S_P}\sum_{j\in S_Q} \overline{b'_{ij}}\, x_{ij} \\
& \textbf{\textit{such that}} && \sum_{j\in S_Q} x_{ij} \le \overline{p'_i}, \quad i \in S_P, \\
& && \sum_{j\in S_P} x_{ij} \le \overline{q'_j}, \quad j \in S_Q, \\
& && x_{ij} \ge 0, \quad (i,j) \in S_P\times S_Q.
\end{aligned}
$$

Transportation Interval Games

For every transportation interval situation (P, Q, B', p', q') and every coalition $S \subset N := P \cup Q$, with origin players $S_P := S \cap P$ and destination players $S_Q := S \cap Q$, and assuming that these sets are both non-empty, we can define the maximization problem of the pessimistic scenario is expressed by:

$$
\begin{aligned}
T(S): \quad & \text{maximize} \quad && \sum_{i \in S_P} \sum_{j \in S_Q} \underline{b'_{ij}} x_{ij} \\
& \text{suchthat} && \sum_{j \in S_Q} x_{ij} \leq \underline{p'_i}, \ i \in S_P, \\
& && \sum_{j \in S_P} x_{ij} \leq \underline{q'_j}, \ j \in S_Q, \\
& && x_{ij} \geq 0, \quad (i, j) \in S_P \times S_Q,
\end{aligned} \qquad (1)
$$

and the maximization problem of the optimistic scenario is stated as:

$$
\begin{aligned}
T(S): \quad & \text{maximize} \quad && \sum_{i \in S_P} \sum_{j \in S_Q} \overline{b'_{ij}} x_{ij} \\
& \text{suchthat} && \sum_{j \in S_Q} x_{ij} \leq \overline{p'_i}, \ i \in S_P, \\
& && \sum_{j \in S_P} x_{ij} \leq \overline{q'_j}, \ j \in S_Q, \\
& && x_{ij} \geq 0, \quad (i, j) \in S_P \times S_Q.
\end{aligned} \qquad (2)
$$

Shapley Value

Definition: Given a coalitional game (N,v), the ***Shapley value*** of player i is given by

$$\phi(i)(N,v) = \frac{1}{N!}\sum_{S\subseteq N\setminus\{i\}}|S|!(N-|S|-1)![v(S\cup\{i\})-v(S)]$$

This captures the *average* marginal contribution of agent i, averaging over all the different sequences according to which the grand coalition could be built up from the empty coalition.

Imagine that the coalition is assembled by starting with the empty set and adding one agent at a time, with the agent to be added chosen uniformly at random.

Interval *Shapley* Value of a Transportation Interval Game

The interval Shapley value $\Phi\colon SMIG^N \to I(\mathbb{R})^N$ is defined by

$$\Phi(v') := \frac{1}{n!}\sum_{\sigma\in\Pi(N)} m^{\sigma}(v'),\ \text{for each } v' \in SMIG^N.$$

The following example shows the calculation of the interval Shapley value in the transportation interval game.

Example. *Consider* $< N, v' >$ *be transportation interval game, where* $N = \{1,2,3\}$ *and the characteristic function* v' *is given by*

$$v'(1) = v'(2) = v'(3) = v'(23) = [0,0],$$
$$v'(12) = [2,4], v'(13) = [2,3], v'(123) = [4,18].$$

Then, the interval marginal vectors are given in the following table. The set of permutations of N are

$$\pi(N) = \begin{Bmatrix} \sigma_1 = (1,2,3), \sigma_2 = (1,3,2), \sigma_3 = (2,1,3), \\ \sigma_4 = (2,3,1), \sigma_5 = (3,1,2), \sigma_6 = (3,2,1) \end{Bmatrix}.$$

Firstly, for $\sigma_2 = (1,3,2)$, we calculate the interval marginal vectors. Then,

$$m_1^{\sigma_2}(v') = v'(1) = [0,0],$$
$$m_2^{\sigma_2}(v') = v'(123) - v'(13) = [4,18] - [2,3] = [2,15],$$
$$m_3^{\sigma_2}(v') = v'(13) - v'(1) = [2,3] - [0,0] = [2,3].$$

The others can be calculated similarly, which is shown in Table 1.

σ	$m_1^{\sigma}(v')$	$m_2^{\sigma}(v')$	$m_3^{\sigma}(v')$
$\sigma_1 = (1,2,3)$	$[0,0]$	$[2,4]$	$[2,14]$
$\sigma_2 = (1,3,2)$	$[0,0]$	$[2,15]$	$[2,3]$
$\sigma_3 = (2,1,3)$	$[2,4]$	$[0,0]$	$[2,14]$
$\sigma_4 = (2,3,1)$	$[4,18]$	$[0,0]$	$[0,0]$
$\sigma_5 = (3,1,2)$	$[2,3]$	$[2,15]$	$[0,0]$
$\sigma_6 = (3,2,1)$	$[4,18]$	$[0,0]$	$[0,0]$

Table 1 illustrates the interval marginal vectors of the cooperative transportation interval game. The average of the six interval marginal vectors is the interval Shapley value of this game, which can be shown as:

$$\Phi(v') \in ([2, \frac{43}{6}], [1, \frac{34}{6}], [1, \frac{31}{6}]).$$

Core

Definition:

A payoff vector x is in the core of a coalitional game (N,v), if and only if

$$\forall S \subseteq N, \sum_{i \in N} x_i = v(N),$$

$$\sum_{i \in S} x_i \geq v(S).$$

So, core elements are imputations which are stable against coalitional deviations. No Coalition can rightfully object to a proposal

$$x \in C(v),$$

because what this coalition is allocated in total according to x at least what it can obtain by splitting off from the grand coaltion. In particular, if

$$\sum_{i \in S} x_i \geq v(S),$$

then in any divison of

$$v(S)$$

among the members of S, at least one player gets strictly less then what he gets according to x.

The interval core of the transportation interval game

The dual problem of the maximization problem of the pessimistic scenario (1) is given by the minimization problem:

$$T^D(S): \quad \text{minimize} \quad \sum_{i \in S_P} \underline{p_i} u_i + \sum_{j \in S_Q} \underline{q_j} v_j$$

$$\text{suchthat} \quad u_i + v_j \geq \underline{b_{ij}}, \quad (i,j) \in S_P \times S_Q,$$

$$u_i, v_j \geq 0, i \in S_P, j \in S_Q,$$

and the dual problem of the maximization problem of the optimistic scenario (2) is given by the minimization problem:

$$T^D(S): \quad \text{minimize} \quad \textstyle\sum_{i \in S_P} \overline{p_i} u_i + \sum_{j \in S_Q} \overline{q_j} v_j$$

$$\text{suchthat} \quad u_i + v_j \geq \overline{b_{ij}}, \ (i,j) \in S_P \times S_Q,$$

$$u_i, v_j \geq 0, i \in S_P, j \in S_Q.$$

Consider the 3-person transportation interval situation (P, Q, B', p', q').

$$P = \{1\}, Q = \{2,3\}, B' = ([1,3] \quad [2,4]), p' = [3,5], q' = ([2,4],[1,3]).$$

The dual problem of the maximization problem of the pessimistic scenario $T^D(\{1,2,3\})$ is:

$$\begin{array}{ll} \textbf{minimize} & \mathbf{3u_1 + 2v_2 + 1v_3} \\ \textbf{suchthat} & \mathbf{u_1 + v_2 \geq 1}, \\ & \mathbf{u_1 + v_3 \geq 2}, \\ & \mathbf{u_1, v_2, v_3 \geq 0}. \end{array}$$

The unique optimal solution of this problem is $(0; 1,2)$. This solution induces the core imputation $(0; 2,2)$. Hovever, the core is

$$C(\underline{v'}) = \{(x_1; y_2, y_3) \in \mathbb{R}^3_+ : x_1 + y_2 \geq 2, x_1 + y_3 \geq 2; x_1 + y_2 + y_3 = 4\},$$

and the dual problem of the maximization problem of the optimistic scenario $T^D(\{1,2,3\})$ is:

$$\begin{array}{ll} \textbf{minimize} & \mathbf{5u_1 + 4v_2 + 3v_3} \\ \textbf{suchthat} & \mathbf{u_1 + v_2 \geq 3}, \\ & \mathbf{u_1 + v_3 \geq 4}, \\ & \mathbf{u_1, v_2, v_3 \geq 0}. \end{array}$$

The unique optimal solution of this problem is $(3; 0,1)$. This solution induces the core imputation $(15; 0,3)$. However, the core is

$$C(\overline{v'}) = \{(x_1; y_2, y_3) \in \mathbb{R}^3_+ : x_1 + y_2 \geq 4, x_1 + y_3 \geq 3; x_1 + y_2 + y_3 = 18\}.$$

Since the transportation interval game is $v' \in \mathcal{J}BIG^N$, then

$$\mathcal{C}(v') = \mathcal{C}^{\blacksquare}(v') = C(\underline{v'}) \blacksquare C(\overline{v'})$$

$$= \{(x_1; y_2, y_3) \in I(\mathbb{R}^3_+) : \{x_1 + y_2 \geq [2,4], x_1 + y_3 \geq [2,3]; x_1 + y_2 + y_3 = [4,18]\}.$$

Conclusion and Outlook

This work studies two-sided transportation situations where the agents' unitary problem parameters (b_{ij}, p_i, q_j) in the transportation model are compact intervals of real numbers.

- Firstly, we introduce the transportation interval situations.
- Secondly, we calculate the interval Shapley value transportation interval game and show that interesting results concerning the interval core of a transportation interval game.
- Moreover, we suggest a procedure that transforms an interval allocation into a payoff vector, under the assumption that only the uncertainty with regard to the value of the grand coalition has been resolved.
- For future research we will study interval semi-infinite transportation problems where supplies and demands are interval numbers. The underlying idea is to consider infinitely divisible goods. One can think of using pipelines instead of containers for the transportation of petrol.

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Thank you very much

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